LOCAL OPTIMALITY CONDITIONS FOR A CONTINUOUS MODEL FOR THE NETWORK ROUTING AND CAPACITY EXPANSION PROBLEM

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Resumo

Considera-se neste trabalho um modelo contínuo para o problema de roteamento e expansão de capacidades cuja função objetivo combina custos de investimento com custos de congestão da rede. Os custos de congestão são modelados por uma função convexa e estritamente crescente para uma dada capacidade. Quando o fluxo num arco atinge um dado ponto de saturação, decide-se por expandir sua capacidade, gerando-se assim uma função de custos nos arcos convexa por partes. Devido a estrutura separável nos arcos da função objetivo, é possível se estabelecer condições de otimalidade local de primeira ordem mesmo sendo a função não diferenciável em toda a parte. A principal contribuição do presente trabalho é a exposição formal de condições de otimalidade local necessárias e suficientes baseadas na não existência de ciclos de custo negativo. Um algoritmo baseado na eliminação de ciclos de custo negativo pode então ser aplicado para se obter soluções localmente ótimas para a o problema de roteamento e expansão de capacidades.

Palavras-Chave: Fluxos Multiproduto, Algoritmo de Anulação de Ciclos, Otimização em Redes, Programação Não Convexa Separável.

Abstract

We consider a continuous model for the routing and capacity expansion problem which objective function represents arc expansion with congestion costs. Congestion is modeled by a convex increasing function for a given capacity and, at a given breakpoint, expansion to a higher capacity is decided, leading to piecewise arc cost functions. Thanks to the separable structure of the cost function, it is possible to put down first-order local optimality conditions even in the presence of breakpoints where the cost function is not differentiable. The main contribution of the present work is the formal exposition of necessary and sufficient local optimality conditions based on the identification of negative-cost cycle for each commodity. An
algorithm based on the elimination of the negative cycles can be then implemented to obtain local optimal solutions of the routing and capacity expansion problem.

**Keywords:** Multicommodity Flows; Cycle Canceling; Network Design; Separable Non Convex Programming

### 1 Introduction

Multicommodity flow problems are known to be very hard to solve even in the simplest situations where flows are continuous and arc costs are linear. The combination of convex costs with discrete decision variables has been considered in the so-called capacity and flow assignment problem and treated with relative success by relaxation or Benders decomposition (see [3], [6], [9]). Skipping the concave situation (see [10]), very few continuous non convex models have been used for network design. Gabrel and Minoux [4] proposed piecewise linear step increasing functions to model the capacity assignment and Luna and Mahey [8] used piecewise convex arc functions to model the routing and capacity expansion problem. In the latter, congestion is modeled by a convex increasing function for a given capacity and, at a given breakpoint, which represents the maximum tolerable congestion for the users, expansion to a higher capacity is decided, decreasing the marginal congestion cost in a discontinuous way.

We will focus here on the continuous model proposed by Luna and Mahey [8] for the routing and capacity expansion problem (CCE) which objective function represents arc expansion with congestion costs. We will assume the following properties of the individual arc congestion functions $\Phi$ throughout the text:

1. $\Phi(c,.)$ is strictly convex, monotone increasing on $(0,c)$
2. $\Phi(c,.)$ is continuously derivable on $(0,c)$ and $\Phi'(c_1,x) < \Phi'(c_0,x)$ for any $0 \leq x < c_0 < c_1$
3. $\Phi(c,0) = 0$ and $\Phi(c,x) \to +\infty$ if $x \to c$

where $c_0$ and $c_1$ are respectively the installed capacity and the capacity for expansion. A well example of such congestion function is Kleinrock’s function $\Phi(c,x) = x/(c – x)$ which express the average delay of a traffic $x$ on an arc with capacity $c$ assuming Poissonian hypotheses for M/M/1 queues (see [2] for example).

The main contribution of the present work is the formal exposition of local optimality conditions for (CCE) based on the identification of negative-cost cycle for each commodity. An algorithm based on the elimination of the negative cycles can then be implemented to obtain local optimal solutions of (CCE).

### 2 A continuous model for capacity expansion

We present in this section the network expansion model. The basic component of the model is a digraph $G = (V,E)$ with $m$ nodes and $n$ arcs representing a communication network. Any kind of traffic between a given pair of nodes $(i,j)$ is treated as a separate commodity $k$. Let $T$ be a given ($n \times n$) traffic requirement matrix such that $t_{ij}$ is the amount of traffic required from origin $i$ and destination $j$.

Given a commodity $k$, we consider a given set of directed paths $P_k$ joining the corresponding origin and destination. This set may be the set of all simple directed paths or a restricted set of feasible paths, for instance with a limited number of hops. Let $\xi_{kp}$ be the amount of flow of commodity $k$ through the path $p \in P_k$ and $a_{kp}$ its arc-path incidence vector defined by
Each component \( x_e \) of the vector \( x \) denotes the total flow on arc \( e \). Then

\[
x_e = \sum_{p \in \mathcal{P}_k} a_{kp} \xi_{kp}.
\]

The set of multicommodity flow vectors, denoted by \( \mathcal{M}(G,T) \) can be described by the implicit arc-path formulation, i.e. for each commodity \( k \) flowing between nodes \( i \) and \( j \) the active paths must satisfy

\[
\sum_p \xi_{kp} = t_{ij}.
\]

We assume now that each arc in the topology is expandable to a capacity \( c_{1e} \) greater than \( c_{0e} \) at a given fixed cost \( \pi_e \). Thus, we can define the (CCE) model that minimize the total congestion cost plus the expansion fixed cost

\[
\begin{align*}
\text{minimize} & \quad \sum_e \min\{\Phi(c_{0e},x_e), \Phi(c_{1e},x_e) + \pi_e\} \\
\text{subject to} & \quad x \in \mathcal{M}(G,T) \\
& \quad x_e \leq c_{1e}, \forall e \in E.
\end{align*}
\]

As shown on the figure below where the non convex resulting arc cost function of (CCE) is represented, we denote by \( \gamma c_{0e} \) with \( 0 < \gamma < 1 \), the breakpoint at which expansion occurs. The parameter \( \gamma \) can thus be interpreted as the relative congestion of an arc beyond which the network manager is willing to pay for expansion. The expansion price converted in congestion cost units is then \( \pi_e = \Phi(c_{0e},\gamma c_{0e}) - \Phi(c_{1e},\gamma c_{0e}) \).

The arc cost function in (CCE) is continuous but non convex and non smooth at the breakpoint \( \gamma c_{0e} \). It was shown by Luna and Mahey [8] how one can easily compute a lower bound on the optimal value of (CCE) by convexifying each arc cost function and summing up the resulting gaps.

\[\begin{array}{c}
f(x_e) \\
\gamma \quad c_0 \quad c_1 \quad x_e
\end{array}\]

2.1 Local optimality conditions

Thanks to the separable structure of the cost function, it is possible to put down first-order local optimality conditions for (CCE) even in the presence of breakpoints where the cost function is not differentiable. Indeed, left and right partial derivatives do exist with respect to all variables. This implies that directional derivatives exist in all directions, allowing to use the first-order conditions for a local minimum: if \( x^* \) is a local minimum of the function \( f \), then \( f(x^*+d) \geq 0 \) in all feasible directions \( d \). We will show below that the convexity of the congestion functions that
build the objective function on each arc not only allows us to characterize that condition using
left and right derivatives but also turns the condition necessary and sufficient.

For any such local optimum let us define

\[ E_0 = \{ e \in E | x^*_e \in [0, \gamma c_0 e) \} \]
\[ E_1 = \{ e \in E | x^*_e \in (\gamma c_1 e, c_1 e) \} \]
\[ E_\gamma = \{ e \in E | x^*_e = \gamma c_0 e \} \]

and let \( g = |E_\gamma| \). Then let partition \( E_\gamma \) in two disjoint subsets of arcs \( E_{\gamma 0} \) and \( E_{\gamma 1} \), so that we can define the corresponding subregion of the feasible set

\[ C_i = \{ x \in M(G,T) | x_e \in [0, \gamma c_0 e] \text{ for } e \in E_0 \cup E_{\gamma 0} \}
\]
\[ x_e \in [\gamma c_0 e, c_1 e] \text{ for } e \in E_1 \cup E_{\gamma 1} \} \]

There are \( 2^g \) such subregions which have disjoint interior points and cover a neighborhood of the feasible set for (CCE). They are defined such that \( x^*_e \in C_i \), for \( i = 1, \ldots, 2^g \). Moreover, the objective function is convex when restricted to any region \( C_i \) and we can write optimality conditions separately in each one of these regions. Indeed, we can associate for each arc in the partition its “active” congestion function, i.e. \( \Phi(c_0 e, x_e) \) for \( e \in E_0 \cup E_{\gamma 0} \) and \( \Phi(c_1 e, x_e) \) for \( e \in E_1 \cup E_{\gamma 1} \), so that \( f(x) \) is simply the sum of the active functions for \( x \in C_i \).

### 2.1 Kuhn-Tucker conditions on set \( C_i \)

There exist multipliers \( u^i_e \) and \( v^i_e \) satisfying

\[ u^i_e = \Phi(c_0 e, x^*_e), \quad 0 < x^*_e < \gamma c_0 e, \quad e \in E_0 \]
\[ u^i_e \leq \Phi(c_0 e, x^*_e), \quad x^*_e = 0, \quad e \in E_0 \]
\[ u^i_e \geq \Phi(c_0 e, x^*_e), \quad x^*_e = \gamma c_0 e, \quad e \in E_{\gamma 0} \]
\[ u^i_e = \Phi(c_1 e, x^*_e), \quad \gamma c_0 e < x^*_e < c_1 e, \quad e \in E_1 \]
\[ u^i_e \leq \Phi(c_1 e, x^*_e), \quad x^*_e = \gamma c_0 e, \quad e \in E_{\gamma 1} \]

and, for all commodity \( k \)

\[ \forall p \in P_k, \quad \text{s.t. } \xi_{kp} > 0, \quad v^k_p = \sum a^\xi_{kp} u^i_e \]
\[ \forall p \in P_k, \quad \text{s.t. } \xi_{kp} = 0, \quad v^k_p \leq \sum a^\xi_{kp} u^i_e \]

Recall that these conditions imply that the active path have minimal lengths with respect to first derivatives of the active functions associated to \( C_i \). The objective function being convex on that region, the conditions are necessary and sufficient. Thus, at a local minimum, these conditions must be satisfied for any of the \( 2^g \) subregions. A crucial question is then to identify situations with the flow blocked at some breakpoint which cannot be optimal. The following theorem shows that when an arc is set to the breakpoint at an optimal solution, it must belong to all active paths for all commodities using that arc.

**Theorem 1**: Suppose that \( x^* \) is a local optimal solution such that \( x^*_e = \gamma c_0 e \) for an arc \( e \), and let \( K_e \) be the set of commodities which flow on arc \( e \). Then all paths carrying flow from all commodities in \( K_e \) contain arc \( e \).

**Proof.** Suppose that arc \( e \) defined by the theorem violates the conclusion; then, there exists at least one commodity \( k \in K_e \) with at least one active path \( p \) which does not contain \( e \). Let \( p \in P_k \)
be a path which contains \( e \) and \( p' \in P_k \) a path which does not. Let now consider two subregions \( C_1 \) and \( C_2 \) associated with the partition of the interval \([0,c_{1e})\) for arc \( e \) in the two sub-intervals \([0,\gamma c_{0e}]\) and \([\gamma c_{0e},c_{1e})\), the remaining arcs being kept equal. Let \( u_1^e \geq \Phi'(c_{0e},x^*_e) \) (resp. \( u_2^e \leq \Phi'(c_{1e},x^*_e) \)) be the multiplier associated with the bound of the active interval of \( C_1 \) (resp. \( C_2 \)). Now considering the length of path \( p' \), it is invariant for the regions \( C_1 \) and \( C_2 \). But this is not the case for path \( p \) as \( u_2^e \leq \Phi'(c_{1e},x^*_e) < \Phi'(c_{0e},x^*_e) \leq u_1^e \). This contradicts the fact that all active paths for a given commodity must have the same (minimal) length for a given \( C_i \) at a local minimum.

We can see with this last result that breakpoints correspond to bottleneck arcs where the total traffic is exactly equal to the breakpoint value, i.e. \( \sum_{k \in Ke} t_k = \gamma c_{0e} \). Thus, any perturbation of one of the demands flowing through arc \( e \) will shift the arc value by the same quantity and consequently get out of the breakpoint. That observation tends to induce the fact that the number \( g \) of breakpoints at a local minimum will remain very low.

### 2.2 Negative cycle optimality conditions

For a given cycle \( \Theta \) of \( G \) and an arbitrary sense of circulation which defines a partition of \( \Theta \) in two subsets of arcs, \( \Theta^+ \) for the direct arcs and \( \Theta^- \) for the reverse arcs, we will use the incident vector \( \theta \in \mathbb{R}^m \) of the cycle with components \( \theta_e \) equal to 1, -1 for the arcs in \( \Theta^+ \), \( \Theta^- \) and 0 for the others.

Let us first recall the classical optimality conditions for the single commodity flow problems with convex arc costs. Given a cycle \( \Theta \) we define its cost for some feasible flow \( x \) by

\[
c(x, \Theta) = \sum_{e \in \Theta^+} f^+_e(x_e) - \sum_{e \in \Theta^-} f^-_e(x_e)
\]

where \( f^+_e(x_e) \) (resp. \( f^-_e(x_e) \)) is the right (resp. left) partial derivative of the cost function \( f \) with respect to \( x_e \). Then the solution is optimal if and only if there does not exist any cycle with negative cost (see [1] for instance). This result has been exploited to build strongly polynomial algorithms in the linear and convex cases (see [6], [7]) but it cannot be extended to multicommodity flow problems in general. However, Ouorou and Mahey [11] have shown that similar optimality conditions and efficient algorithms can be derived for smooth strictly convex congestion functions. We will show below that a negative cycle condition for each commodity characterizes a local minimum of (CCE) even in the presence of breakpoints and it works thanks to the properties of the congestion functions.

Let \( x \) be a feasible solution of (CCE). We will call a cycle \( \Theta \) \( k \)-feasible if it presents a strictly positive residual, i.e. if we can augment the commodity flow value \( x_k^e \) on the direct arcs of \( \Theta \) and reduce these values on the reverse arcs. In our model, a \( k \)-feasible cycle is such that all reverse arcs carry a positive value of commodity \( k \).

**Theorem 2:** A feasible solution \( x^* \) is a local minimum of (CCE) if and only if, for all commodities \( k = 1, ..., K \), there does not exist any \( k \)-feasible cycle with negative cost.

**Proof.** Suppose first that, for some feasible \( x \), there exists a negative cost \( k \)-feasible cycle \( \Theta \). Then there exists \( \varepsilon > 0 \) such that \( y = x + \varepsilon \theta \) is still feasible. But the directional derivative of the cost function \( f \) in the direction \( y - x \) is simply \( f'(x,\theta) = c(x, \Theta) \) which is negative. Thus \( x \) cannot be a local minimum.

Let \( x^* \) be a feasible solution of (CCE) such that \( G \) contains no negative-cost \( k \)-feasible cycle. We will prove that, for each subregion \( C_i \) as defined above, there exist multipliers which satisfy the first-order optimality conditions. To compute these multipliers, we define a convex
multicommodity flow problem \((P_i)\) associated with \(C_i\) by fixing the arc capacities to \(c_0e\) if \(e \in E_0 \cup E_{\gamma 0}\) and to \(c_1e\) if \(e \in E_1 \cup E_{\gamma 1}\); the corresponding congestion functions are fixed to \(\Phi(c_0e, x_e)\) (resp. \(\Phi(c_1e, x_e)\)) for each arc of the partition. To simplify the notation, we denote by \(q_e(x_e)\) the congestion function of arc \(e\) in the convex problem \((P_i)\).

We can thus apply the optimality condition for convex smooth congestion functions obtained in [11]: if there are no negative cost \(k\)-feasible cycle, \(x^*\) is optimal for the convex routing problem. We must just verify that the absence of negative cycle with the cost defined by \(c(x, \Theta) = \sum_{e \in \Theta^+} f^+(x_e) - \sum_{e \in \Theta^-} f^-(x_e)\) implies the absence of negative cycle with the derivatives of the convex functions \(q_e\). Indeed, for any \(k\)-feasible cycle, we have

\[
\begin{align*}
q_e'(x^*_e) &\geq f^+(x^*_e) \quad \forall e \in \Theta^+ \cap E_{\gamma 0} \\
q_e'(x^*_e) &= f^+(x^*_e) \quad \forall e \in \Theta^+ \cap E_{\gamma 1} \\
q_e'(x^*_e) &= f^-(x^*_e) \quad \forall e \in \Theta^- \cap E_{\gamma 0} \\
q_e'(x^*_e) &\leq f^-(x^*_e) \quad \forall e \in \Theta^- \cap E_{\gamma 1}
\end{align*}
\]

and this implies that \(\sum_{e \in \Theta^+} q_e'(x_e) - \sum_{e \in \Theta^-} q_e'(x_e) \geq c(x, \Theta) \geq 0\). Thus, \(x^*\) is an optimal solution for the routing problem \((P_i)\) and we can define optimal multipliers \(u_e^i\) and \(v_k\) such that

\[
\begin{align*}
& u_e^i = q_e'(x^*_e) \quad \text{if } x^*_e > 0 \\
& u_e^i \leq q_e'(x^*_e) \quad \text{if } x^*_e = 0 \\
& v_k^i = \sum_p a_{kp}^i u_{e}^i \quad \text{if } \xi_k > 0 \\
& v_k^i \geq \sum_p a_{kp}^i u_{e}^i \quad \text{if } \xi_k = 0
\end{align*}
\]

We will now verify that these multipliers satisfy the Kuhn-Tucker optimality conditions for a local minimum of \((CCE)\) with respect to the subregion \(C_i\). Again, we only need to test the conditions at arcs where the flow has its breakpoint value. Indeed, we verify

\[
\begin{align*}
& u_e^i = q_e'(x^*_e) = f^+(x^*_e) \quad \forall e \in E_{\gamma 0} \\
& u_e^i = q_e'(x^*_e) = f^-(x^*_e) \quad \forall e \in E_{\gamma 1}
\end{align*}
\]

and this proves that \(x^*\) is optimal for \((CCE)\) restricted to the subregion \(C_i\), thus, as this construction can be repeated for all subregions, \(x^*\) is a local minimum for \((CCE)\). \(\square\)

Observe that the key fact which leads to the proof of the sufficient condition of the precedent theorem are the inequalities that bound the reduced costs of a cycle. It works because, at the breakpoints, \(f^+(x_e) < f^-(x_e)\) and the result cannot be extended to a convex non smooth congestion function as already observed in [11].

3 Conclusions

We have established necessary and sufficient local optimality conditions for the continuous model proposed by Luna and Mahey [8] for the network routing and capacity expansion problem. Our result, based on the flow structure of a feasible solution, allows the use of negative-cost cycle canceling methods (see [6], [11]) to obtain local optimum solutions. Work in progress includes computational experiments to study the numerical behavior of such an algorithm for the network routing and capacity expansion problem.

References


