CHI-SQUARE CONTROL CHART FOR LINEAR PROFILES

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ABSTRACT
The control chart is useful in the monitoring of a production process by keeping an eye in the key quality characteristics of the outcome of the production process. The monitoring can be accomplished by means of univariate or multivariate charts. In some cases, the key characteristics of the production process can be represented by some profile, that is, a linear or non-linear relationship between some of its variables. Here we consider the monitoring of a linear profile where the estimatives of the parameters of the model are obtained by Kalman filter and also by the usual least square method. The $\chi^2$ control chart is used to monitor the process. By a detailed simulation, we conclude that the chart based on the Kalman filter estimatives presents a better performance than the chart based on the ordinary least square estimatives. This is however expected since the comparison between a chart based on a statistic that takes into account only present values and another that also takes into account past values must favor the second.

KEYWORDS. Statistical process control, Monitoring of linear profiles, Kalman filter. Applications to Industry.

RESUMO
O gráfico de controle é útil no monitoramento de um processo de produção ao levar em consideração as características de qualidade mais relevantes da saída do processo de produção. O monitoramento pode ser realizado por meio de gráficos univariados ou multivariados. Em alguns casos, as principais características de qualidade podem ser representadas por algum perfil, isto é, uma relação linear ou não-linear entre algumas das suas variáveis. Aqui consideramos o monitoramento de um perfil linear, onde as estimativas dos parâmetros do modelo são obtidas por meio do filtro de Kalman e pelo método usual de mínimos quadrados. O gráfico de controle utilizado é o $\chi^2$. Através de uma simulação, concluímos que o gráfico de controle baseado nas estimativas do filtro de Kalman apresenta melhor desempenho, o que é esperado uma vez que usa mais informações passadas do que o baseado nas estimativas de mínimos quadrados ordinares.

1. Introduction

Since its introduction by W. A. Shewhart in 1920, control charts have been used in industry, service sectors and more, recently, in healthcare establishments, in the monitoring of various processes in order to avoid undesired results.

These charts can be of the univariate type, as for example when a mean or a dispersion of a characteristic is been monitored, or the multivariate type, when two or more related characteristics, represented by a vector, are under surveillance (Montgomery, 1991).

Lately several investigation on monitoring of processes has been carried out, as for instance, control charts for the mean and/or the variance (Rahim, 1989, Costa, 1993, 1998, 1999, Costa and Rahim, 2004, De Magalhães and Moura Neto, 2005, De Magalhães et al., 2006, 2009); multivariate control charts based on Hotelling’s $T^2$ statistics and $\chi^2$ statistics (Kourti and MacGregor, 1996, Mason et al., 1995); monitoring of processes when the quality characteristic is represented by a linear profile or model (Woodall et al., 2004, Kang and Albin, 2000).

In their investigation of the monitoring of a linear profile, Kang and Albin (2000) proposed that, at each time interval, a sample of size $n$ was extracted from the process, and an estimate of the coefficients of a linear regression model of the profile was computed, as well as, a $T^2$ statistics and the use a control chart for such statistic. They also proposed joint control charts for the range and the exponentially weighted moving average (EWMA/R), for the monitoring of the residuals. These two charts were applied to the monitoring of mass flux controllers (MFC), knowing, in advance that in a MFC, there is an approximate linear relation between the measured pressure in the chamber and the level of the flux of the gas. By means of a simulation, they compared the performance of the charts by computing the average number of samples until a signal, and concluded that EWMA/R chart is faster in the detection of small changes in the parameters of the MFC, supposedly because that chart takes into account past information, and for large changes in the parameters, both charts operate well.

This article investigates the monitoring of processes when the quality characteristic of the process is best represented by a linear profile. The estimation of the regression parameters is done by least squares method and by the Kalman filter. A $\chi^0_2$ statistic is computed for each sample that is draw from the process. This study shows that it is a great advantage to use the control chart based on the Kalman filter estimatives than the chart based on the ordinary least square estimatives. Further investigation is required to fully access the performance of this methodology. Nonetheless, the results presented in this paper are very convincing. The simulation considers only parametric changes at the beginning of observation time. In such a case, the comparison between the chart based on a statistic that takes into account only present values and another that also takes into account past values must favor the second.

2. Estimation of the Coefficients in a Linear Regression

Let $Y$ denote a quality characteristic that can be expressed approximately by a linear function, in the following way

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_r X_r + \epsilon$$

where $\beta_l$ for $l = 0, \ldots, r$ are constants, $X_r$ are independent variables of the process and $\epsilon$ is a normal random variable with zero mean.

Geometrically this model can be interpreted in the following way. This problem can be treated in the following way. Let a matrix $X_{nxp}$ with $p=r+1$ columns, and a random vector $y_{(n1)}$. If $y$ does not belong to the column space of $X$, there is no linear combination of the columns of $X$ with coefficients ($\beta_0, \ldots, \beta_r$) that equals that vector. In other words, there is no solution to
the linear system of equation, \( y = X\beta \). Therefore, one can introduce the non-deterministic model given by,

\[
y = X\beta + \epsilon ,
\]

where the mean of \( y \) is \( \text{E}(y) = X\beta \), \( \epsilon \sim N(0, V) \) is the error random vector, with \( V = E(\epsilon \epsilon^T) = \sigma^2 I \), and \( \sigma^2 \), the variance of the components \( \epsilon_j \) of the error vector.

As an example, of an application of this model we mention briefly the monitoring of mass flux controllers (MFC) considered by Kang and Albin (2000): the pressure measured in a chamber is, approximately, a linear function of the level of the fluxes of a gas. In the probabilistic model, the measured pressure in the chamber is represented by the dependent random variable \( Y \), while the levels of the flux of gas are represented by an independent variable \( X \). Usually, the vector of parameters \( \beta \) is estimated by the least squares method. Here \( \beta \) will also be estimated by the Kalman filter. The next two sections present those two estimation techniques for the parameters of the regression model.

2.1 Least Squares Method

The least squares solution (\( \hat{\beta} \) estimator), which minimizes the error \( \epsilon = \|X\beta - y\| \), is the point \( q \) in the column space of \( X \) nearest to \( y \). By geometry, \( q \) must be the projection of \( y \) in the column space of \( X \). The error vector, whose size is minimum, must be perpendicular to the column space.

The set of vectors perpendicular to the column space belong to the left null space of \( X \), which is the null space of \( X^T \). Therefore, the vector \( y - X\hat{\beta} \) belongs to the null space of \( X^T \). If \( X \) has linearly independent columns, then \( X^T \) is invertible, and in this case, one can obtain the vector of estimates of the vector of parameters, by the least squares method:

\[
X^T (y - X\hat{\beta}) = 0 \Rightarrow X^T X\hat{\beta} = X^T y \Rightarrow \hat{\beta} = (X^T X)^{-1} X^T y .
\]

(2)

By the linear combination theorem, the linear combination of independent normal random variables is itself normal. Since \( y \) is a random vector whose components are independent normal random vector \( \hat{\beta} \) is also a normal multivariate random variable, since it is the linear combination of the components of \( y \) taking the coefficients from the columns of the matrix \( (X^T X)^{-1} X^T \). The estimator \( \hat{\beta} \) is unbiased for the parameter \( \beta \) since its mean is given by:

\[
\text{E}(\hat{\beta}) = \text{E}((X^T X)^{-1} X^T y) = (X^T X)^{-1} X^T \text{E}(y) = (X^T X)^{-1} X^T X\beta = \beta .
\]

(3)

By substituting equation (1) in (2) we find:

\[
\hat{\beta} = (X^T X)^{-1} X^T (X\beta + \epsilon) = \beta + (X^T X)^{-1} X^T \epsilon .
\]

(4)

From equation (4) we then get:
\[ \hat{\beta} - \beta = (X^T X)^{-1} X^T \epsilon. \] 

(5)

By definition and from equation (5), the covariance matrix of \( \hat{\beta} \):

\[ \Sigma = E \left[ (\hat{\beta} - E(\hat{\beta})) (\hat{\beta} - E(\hat{\beta}))^T \right] = E \left[ (X^T X)^{-1} X^T \epsilon \epsilon^T (X^T)^T \right] = 
\]

\[ = (X^T X)^{-1} X^T X (X^T X)^{-1} \Sigma = (X^T X)^{-1} \sigma^2 \]

(6)

It can be proven that the estimators obtained by least squares have minimum variance. The distribution of the estimator of the parameter \( \hat{\beta} \) is given by \( \hat{\beta} \sim N_p (\beta, \Sigma) \).

When the measurement errors are not equally trusted, one can use weighted least squares where the weights are defined by the inverse of the covariance matrix, \( V \), of the measurement errors, \( \epsilon_j \), in this way, giving more weight to those measurements that have smaller variance of the error. In this particular case, the estimator of parameter \( \hat{\beta} \) is given by

\[ \hat{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} y, \]

and its covariance matrix is

\[ \Sigma = (X^T V^{-1} X)^{-1}, \]

where \( V_{ij} = E(\epsilon_i \epsilon_j) \) are the error variances, and \( V_{ik} = E(\epsilon_i \epsilon_k) \), their covariances.

2.2 Kalman Filter

In some problems, at each instant \( t=m \), one gets a new information in the form of a point in \( \mathbb{R}^2 \), \( y_{(m)} \). After adjusting a straight line to the set of points available at time \( t=m \), a new information arrives coming from a new sample extracted from the process, \( y_{(m+1)} \). One then would like to modify the coefficients of the straight line, in order to take into account the new information and to get a better fitting line, but would like to do so in such a way to avoid to have to do all the work from the beginning. One can get the coefficients of the new adjusted line recursively (recursive least squares method), by solving the combined system:

\[ y^{(0)} = X \beta + \epsilon^{(0)}, \ y^{(i)} = X \beta + \epsilon^{(i)}, \ ... \ y^{(m)} = X \beta + \epsilon^{(m)}. \] 

(7)

Moreover, it can happen that that there is a dynamic law that the straight line satisfies. Here, we take a linear dynamic model,

\[ \beta^{i+1} = A \beta_i + w_i. \]

The solution to this non-stationary problem is given by Kalman filter,

\[ X \beta_i = y_i \quad \text{(with error} \ \epsilon_i) \]

(8)
\[-A\beta^i + \beta^{i+1} = 0\]  \hspace{1cm} \text{(with error } w^i) \hspace{1cm} (9) \]

where \(A\) is known and the random variable \(w^i \sim N(0, Q)\) represents a modeling error of the process.

For the first iteration, having been entered the data vectors \(y^{(0)}\) and \(y^{(1)}\), the combined system is:

\[
\begin{bmatrix}
X & cI & cI \\
X & cI & cI
\end{bmatrix}
\begin{bmatrix}
\hat{\beta}^{(0)} \\
\hat{\beta}^{(1)}
\end{bmatrix}
= 
\begin{bmatrix}
y^{(0)} \\
y^{(1)}
\end{bmatrix},
\hspace{1cm} (10)
\]

where \(c\) is the weight attributed to the dynamic equation (9).

The solution of the combined system, equation (10), can be found by least squares. In the first step, one determines the estimate for \(\beta\) and call it \(\hat{\beta}^{(1/1)}\); for the second step, one augments the matrix by a column and two block lines and computes the estimate \(\hat{\beta}^{(2/2)}\). In each succeeding step, step \(i\), one augments the matrix by a column and two block lines, determining the estimate \(\hat{\beta}^{i/1}\).

The estimate \(\hat{\beta}^{i/1}\) is called filtered value of \(\beta\). Using it and making use of the dynamic equation, one gets the prediction for the next time, \(\hat{\beta}^{i+1/1} = A \hat{\beta}^{i/1}\). When new observations come in, \(y^{i+1}\), the predicted value, \(\hat{\beta}^{i+1/1}\), is substituted by the filtered value, \(\hat{\beta}^{i+1/1+i}\). It is not reasonable to think that \(y^{i+1}\) is equal to its prediction, \(X\beta^{i+1/1}\). In fact, usually, \(y^{i+1} - X\beta^{i+1/1} \neq 0\). Also, \(\hat{\beta}^{i+1/1+i} - \hat{\beta}^{i+1/1} \neq 0\). By Kalman filter, the system given above is solved recursively, see Strang (2007).

One reference to Kalman filter is Welch e Bishop (2006) from where the following prediction and filtering equations have been taken:

Prediction or actualization equations of Kalman Filter:

\[
i) \quad \hat{\beta}^{i+1/1} = A \hat{\beta}^{i/1}; \hspace{1cm} (11)
\]

\[
ii) \quad P^{i+1/1} = AP^iA^T + Q; \hspace{1cm} (12)
\]

Measurement actualization or filtering equations:

\[
iii) \quad K^{i+1} = P^{i+1/1}X^T(XP^{i+1/1}X^T + V)^{-1}; \hspace{1cm} (13)
\]

\[
iv) \quad \hat{\beta}^{i+1/1+i} = \hat{\beta}^{i+1/1} + K^{i+1}(y^{i+1} - X\hat{\beta}^{i+1/1}); \hspace{1cm} (14)
\]

\[
v) \quad P^{i+1} = (I - K^{i+1}X)P^{i+1/1}. \hspace{1cm} (15)
\]

where the covariances of the prediction error \(\beta^{i+1} - \hat{\beta}^{i+1/1}\) and of the estimator error are given by:

\[
E\left[ (\beta^{i+1} - \hat{\beta}^{i+1/1})(\beta^{i+1} - \hat{\beta}^{i+1/1})^T \right] = P^{i+1/1}; \hspace{1cm} (16)
\]

\[
E\left[ (\beta^i - \hat{\beta}^{i/1})(\beta^i - \hat{\beta}^{i/1})^T \right] = P^i. \hspace{1cm} (17)
\]
In the updating in time, one computes the prediction of the estimative for time $i+1$, and computes the covariance matrix of the prediction error, and in the updating of the measurements, one computes first the gain matrix, $K^{i+1}$, and through a new set of data, $y^{i+1}$, one computes the filtered value and the covariance matrix of the estimator error, $P^{i+1}$, for time $i+1$. It can be proven that the Kalman filter provides a best linear unbiased estimator (see Strang, 2007).

3. Control Charts for Profiles

The main objective of a control chart for the linear profile is to monitor the parameters of the model, given in equation (1), that is, the regression coefficients, $\beta_0, ..., \beta_r$.

If the parameters $\beta$ and $\sigma^2$ of the model are known, when the process is operating in its stable, adjusted, normal way, it is not necessary to go through their estimation, the so-called phase I of control charts methodology. One goes immediately to phase II, corresponding to the on-line use of the control chart to monitor the production process. In this phase, one periodically extracts randomly a sample from the process, tagged $i$, of size $n$, compute the estimates of the regression coefficients and compute the chosen statistics to be plotted on the $\chi^2_p$ chart, for each sample, in the following way,

$$\chi^2_0 = \left(\hat{\beta}^i - \beta\right)^T \Sigma^{-1} \left(\hat{\beta}^i - \beta\right),$$

(18)

where $\hat{\beta}^i = \left[\hat{\beta}_{0}^i \ldots \hat{\beta}_{r}^i\right]$, $\beta = \left[\beta_0 \ldots \beta_r\right]$, and $\Sigma = \left[\begin{array}{ccc} \sigma^2_0 & \sigma^2_{01} & \cdots & \sigma^2_{0r} \\ \sigma^2_{01} & \sigma^2_1 & \cdots & \sigma^2_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2_{0r} & \sigma^2_{1r} & \cdots & \sigma^2_r \end{array}\right] = (X^T X)^{-1} \sigma^2$

are, respectively, the estimator, for the $i$ sample, of the parameter $\beta$, the mean vector and its covariance matrix.

In the chart, the upper control limit, $UCL$, is given by $UCL = \chi^2_{p, \alpha}$, where $\alpha$ is the type I error probability, that is, the probability that the chart signals when the process is under control, in short, a false alarm.

If all the components of $\beta$, the coefficients $\beta_0, ..., \beta_r$, remain on their under control values, the values of the $\chi^2_0$ statistics will be less than $UCL$. However, if at least one of the coefficients change for a new value, the probability of $\chi^2_p$ being greater than $UCL$ increases.

If the points plotted on the chart stay below the $UCL$, and exhibit a random pattern, the chart indicates that the process is under control, and that it do not needs intervention. However, if this is not the case, the chart recommends intervening in the process, because it has signaled that it may be out of control. A search for a special cause of variation should be undertaken and the production process should be accordingly adjusted or repaired.

In brief: periodically one extracts a sample of the production process, compute and plot $\chi^2_0$, and test the hypothesis $H_0$: $\beta$ (vector of coefficients) is under control; or $H_1$: $\beta$ is out of control.

One of the performance measures of control charts is the average number of samples till an alarm, $ANS$ (Montgomery, 1991), whether the process is in-control or out-of-control. In this work, $ANS$ gives, in average, the number of samples needed till that a statistic $\chi^2_0$ hits above the $UCL$. 

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A qualitative requirement on ANS is that it is small when the process is out-of-control and that is large otherwise.

It can be proven that the statistics $\chi^2_o$ has a $\chi^2$ distribution, with $p=r+1$ degrees of freedom. The $\chi^2$ distribution is defined by means of a standard normal distribution. If $Z \sim N(0,1)$, then $Z^2 \sim \chi^2_1$ ($\chi^2$ distribution with one degree of freedom). In general, consider the random vector $\hat{\beta} \sim N_p(\beta, \Sigma)$ and let $B$ be a matrix such that $BB^T = \Sigma$. Then, $B^{-1}(\hat{\beta} - \beta) \sim N_{p}(0, I)$. If $\Sigma$ is non-singular, $\Sigma^{-1}$ exists, and the $\chi^2$ distribution with $p$ degrees of freedom is given by:

$$\chi^2_p = (\hat{\beta} - \beta)^T \Sigma^{-1} (\hat{\beta} - \beta).$$

### 4. Simulation of the Average Number of Samples until a Signal

Our aim in this section is to simulate the average number of samples needed until a point, $\chi^2_0$, falls above the upper control limit, $UCL = \chi^2_{p, \alpha}$.

To determine the number of samples until a signal we proceed in the following way. Let $N = \sum_{i=1}^{m} C(i)$ be the number of times the control chart signals, where $C(i)$ is the number of times the signal occurs in the $i$-th sample. Therefore, an estimate for the average number of samples until a signal is given by

$$ANS = \frac{\sum_{i=1}^{m} i C(i)}{N}.$$

The simulation of ANS is done by the following algorithm.

**Algorithm**

i) simulate a $(nx1)$ vector of measurements, by $y = X\beta + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2)$ for $1 \leq j \leq n$;

ii) estimate the parameter $\hat{\beta}$ by either the least squares method or by Kalman filter;

iii) for that sample, compute and plot on the chart the statistics $\chi^2_0$;

iv) repeat the steps (i)-(iii) until a point falls above the $UCL = \chi^2_{p, \alpha}$ (until an out-of-control signal);

v) repeat (i)-(iv) a number of times and count the number of times the signal occurs at sample $i$, for several values of $i$, determining $C(i)$;

vi) finally compute ANS.

### 5. Analysis of the Results and Conclusions

A MATLAB routine was written and with it simulations where performed, with least squares method (LSM) and with Kalman filter. The results, presented below, represent the average number of samples until a signal when the process is in control and when it is out-of-control. For all simulation the number of samples was 100.

We consider the example presented by Kang and Albin (2000). For the regression model $y = X\beta + \varepsilon$, the following values were chosen:
\[
X = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
0 & 1 & \cdots & 1
\end{pmatrix}^T; \quad \beta = \begin{pmatrix} 3 \\ 2 \end{pmatrix}; \quad \epsilon_j \sim N(0,1), \text{ and } Q = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}
\]

This last matrix is the covariance of the error \( w^i = \beta^{i+1} - A \beta^i \) in the Kalman filter, and \( \rho \), the covariance of the components of the error vector. Then \( (\hat{\beta}^i - \beta)^T \Sigma^{-1} (\hat{\beta}^i - \beta) \sim \chi^2_2 \), and, if one sets the number of false alarms at \( \text{ANS} = 200 \), then the \( \text{UCL} = 10,597 \). Here, when the process is in control, \( \beta = [\beta_0 + \delta_0 \sigma, \beta_1 + \delta_1 \sigma]^T \) with \([\delta_0, \delta_1]^T = 0\). Moreover, the dynamic matrix \( A \), in equation (9), is chosen equal to the identity because, if the process is in control, one expects that, at most, its parameters only vary randomly, with a small variance, as accounted for by \( w^i \), while \( A \) is responsible for a systematic change in the parameters, unless it is set equal to the identity matrix.

Figure 1 presents a comparison between the simulations of \( \text{ANS} \) when \( \beta_0 \) changes to \( \beta_0 + \delta_0 \sigma \), with estimates defined by least squares (Fig.1 a) and with the estimates computed by Kalman filter, when \( \sigma^2 = 1 \), the assumed true variance of the measurement error, and when the variance diminishes to \( \sigma^2 = 0.9 \). The results of the simulations are presented in Table 1 for the case \( \sigma^2 = 0.9 \).

Figure 2 presents a comparison between the simulations of \( \text{ANS} \) when the coefficient \( \beta_1 \) changes to \( \beta_1 + \delta_1 \sigma \), also considering estimates done by least squares and by Kalman filter, when \( \sigma^2 = 1 \) and \( \sigma^2 = 0.9 \). Table 2 presents the results of these simulations for \( \sigma^2 = 0.9 \).

Figure 1. Average number of samples until a signal when \( \beta_0 \) changes to \( \beta_0 + \delta_0 \sigma \), with \( \rho = 0, \rho = 0.25, \rho = 0.5 \) and \( \rho = 0.75 \).

a) \( \sigma^2 = 1 \); b) \( \sigma^2 = 0.9 \).
For Kalman filter, a change in the variance from $\sigma^2 = 1$ to $\sigma^2 = 0.9$, has the effect of increase in the “belief” of the measurements (they are more precise), and therefore, the it is more sensitive to the estimates of $\hat{\beta}$ used in the computation of $\chi^2$. As a consequence, the number of samples until a signal and its mean, ANS, when $\sigma^2 = 0.9$ are smaller, and when this happens, the performance of the control chart is better when the estimates of $\beta$, $\hat{\beta}$ are determined by the Kalman filter instead of the least squares method.

Table 1. Average number of samples until a signal when $\beta_0$ changes to $\beta_0 + \delta_0 \sigma$ with $\rho = 0; 0.25; 0.5; 0.75$ and $\sigma^2 = 0.9$.

<table>
<thead>
<tr>
<th>$\delta_0$</th>
<th>LSM</th>
<th>Kalman filter $\rho = 0$</th>
<th>Kalman filter $\rho = 0.25$</th>
<th>Kalman filter $\rho = 0.5$</th>
<th>Kalman filter $\rho = 0.75$</th>
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Table 2. Average number of samples until a signal when $\beta_1$ changes to $\beta_1 + \delta_i \sigma$, when $\rho = 0; 0.25; 0.5; 0.75$ and $\sigma^2 = 0.9$.

<table>
<thead>
<tr>
<th>$\delta_i$</th>
<th>LSM</th>
<th>Kalman filter $\rho = 0$</th>
<th>Kalman filter $\rho = 0.25$</th>
<th>Kalman filter $\rho = 0.5$</th>
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<td>1.62</td>
</tr>
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<td>1.21</td>
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<td>1.26</td>
<td>1.23</td>
<td>1.17</td>
</tr>
<tr>
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<td>1.11</td>
<td>1.07</td>
<td>1.11</td>
<td>1.09</td>
<td>1.06</td>
</tr>
<tr>
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<td>1.03</td>
<td>1.04</td>
<td>1.07</td>
<td>1.03</td>
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</table>

The results are presented in Tables 3 and 4. The gain, in percentage of ANS, for the estimates computed by Kalman filter compared with those computed by least squares, is presented.

Table 3. Percentage gain in ANS when the parameters of the model are estimated by Kalman filter compared with the estimates done by the least squares method, for changes in $\beta_0$, and in values of $\rho$.

<table>
<thead>
<tr>
<th>$\delta_0$</th>
<th>$\rho = 0$</th>
<th>$\rho = 0.25$</th>
<th>$\rho = 0.5$</th>
<th>$\rho = 0.75$</th>
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<tbody>
<tr>
<td>0</td>
<td>5.57</td>
<td>9.31</td>
<td>8.68</td>
<td>-2.37</td>
</tr>
<tr>
<td>0.2</td>
<td>33.98</td>
<td>18.14</td>
<td>28.26</td>
<td>11.78</td>
</tr>
<tr>
<td>0.4</td>
<td>42.98</td>
<td>29.11</td>
<td>18.38</td>
<td>28.94</td>
</tr>
<tr>
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<td>24.41</td>
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<td>14.64</td>
</tr>
<tr>
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<td>1.03</td>
<td>2.74</td>
<td>7.88</td>
</tr>
<tr>
<td>1</td>
<td>8.38</td>
<td>1.12</td>
<td>10.61</td>
<td>10.61</td>
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<tr>
<td>1.2</td>
<td>10.61</td>
<td>9.09</td>
<td>7.58</td>
<td>7.58</td>
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<tr>
<td>1.4</td>
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<td>-0.92</td>
<td>-0.92</td>
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<td>-1.00</td>
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<td>19.35</td>
<td>19.35</td>
<td>19.35</td>
<td>19.35</td>
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</table>
Table 4. Percentage gain of Kalman filter with respect to least squares when changes are made to \( \beta_1 \), for different values of \( \rho \).

<table>
<thead>
<tr>
<th>( \delta_i )</th>
<th>( \rho = 0 )</th>
<th>( \rho = 0.25 )</th>
<th>( \rho = 0.5 )</th>
<th>( \rho = 0.75 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(%)</td>
<td>(%)</td>
<td>(%)</td>
<td>(%)</td>
<td>(%)</td>
</tr>
<tr>
<td>0</td>
<td>12.06</td>
<td>21.91</td>
<td>12.29</td>
<td>-5.38</td>
</tr>
<tr>
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<td>-19.38</td>
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<td>20.92</td>
<td>22.42</td>
<td>21.97</td>
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<tr>
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<td>19.49</td>
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<td>28.76</td>
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<td>0.125</td>
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<td>20.61</td>
<td>25.06</td>
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<tr>
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<td>11.73</td>
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</tr>
<tr>
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<td>3.31</td>
</tr>
<tr>
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<td>0.00</td>
<td>1.80</td>
<td>4.50</td>
</tr>
<tr>
<td>0.25</td>
<td>0.96</td>
<td>0.00</td>
<td>-2.88</td>
<td>0.96</td>
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</table>

Tables 3 and 4 show the gain, in percentage of ANSI, for estimates done by means of Kalman filter instead of least squares method. The use of estimates, based on the Kalman filter, is considerable better, especially when shifts in the mean and variance are small.

In the monitoring of a linear profile, one can conclude that an \( \chi^2 \) chart is faster in detecting small to moderate shifts in the parameters, whenever the estimation is done by means of the Kalman filter. Moreover, since applications of Kalman filter to control charts, as far as we are aware of, has not been done so far, we expect that it can give very interesting results, and may contribute to the construction of better, more stable, control charts, even if the filter in itself is quite demanding on theory.

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Costa, A.F.B. and Rahim, M.A. (2004), Joint \( \bar{X} \) and \( R \) charts with two stage samplings, *Quality and Reliability Engineering International*, 20, 699-708.


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