Option Pricing under a Nonlinear and Nonnormal GARCH

Renato Costa

Department of Electrical Engineering, PUC-Rio; E-mail: renato@ele.puc-rio.br

Álvaro Veiga

Department of Electrical Engineering, PUC-Rio; E-mail: alvf@ele.puc-rio.br

Tak Kuen Siu

Department of Actuarial Studies, Faculty of Business and Economics,

Macquarie University; E-mail: Ken.Siu@efs.mq.edu.au

June 30, 2010

Abstract

We investigate the pricing of options in a class of discrete-time Flexible Coefficient Generalized Autoregressive Conditional Heteroskedastic (FC-GARCH) models with non-normal innovations. A conditional Esscher transform was used to select a price kernel for valuation in the incomplete market. We provide a numerical study on the pricing results when the GARCH innovations have a normal distribution or a shifted-Gamma distribution and identify some key features of the pricing results.

Keywords: Option Pricing; GARCH Models; Conditional Esscher Transform. Statistics.

§1. Introduction

In this paper, we study the option valuation problem in a class of discrete-time Flexible Co-efficient Generalized Autoregressive Conditional Heteroskedastic (FC-GARCH) models with non-normal innovations. The original version of this class of models was introduced by Veiga and Medeiros(2008) for the case of normal innovations. The class of FC-GARCH models can incorporate various empirical features of asset's returns, such as the asymmetric impact of past returns on conditional volatilities and long-memory effect of conditional volatility. It also nests a number of important ARCH-type models in the literature. Here we investigate the use of the conditional Esscher transform for option valuation under this class of models and explore its pricing implications. Simulation studies are conducted to illustrate the practical implementation of the proposed model and to document some features of option prices arising from the proposed model.

The rest of the paper is structured as follows. The next section presents a FC-GARCH model with non-normal innovations for modeling the asset returns. In Section 3, we discuss the valuation approach based on the conditional Esscher transform. Section 4 studies some parametric cases of the model. Section 5 gives the simulation results and some discussion on the results. The final section summarizes the main finding of the paper.

§2. Flexible Coefficient Generalized Autoregressive Conditional Heteroskedastic (FC-GARCH) models for Asset Returns

We consider a discrete-time economy with a bond B and a a share S. Let $\mathcal{T}$ denote the time index set $\{0, 1, 2, \ldots, T\}$ of the economy. To model uncertainty, we fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where $\mathcal{P}$ is a real-world probability measure. To simplify our analysis, we assume that the continuously compounded rate of interest from the bond is a constant, say r per period. Consequently, the bond-price process $\{B_t | t \in \mathcal{T}\}$ evolves over time as:

$$B_t \;=\; B_{t-1}e^r \;, \quad B_0 = 1 \;. \tag{2.1}$$

Let $\epsilon = \{\epsilon_t\}_{t \in \mathcal{T}}$ be the return innovations of the share S, where we take $\epsilon_0 = 0$ by convention. Suppose $\{\epsilon_t | t \in \mathcal{T} \backslash \{0\}\}$ are independent and identically distributed, (i.i.d.), with common distribution $D(0, 1)$, where $D(0, 1)$ represents a general distribution with zero mean and unit variance.

Let $S := \{S_t\}_{t \in \mathcal{T}}$ be the price process of the share S. Let $Y_t := \ln(S_t/S_{t-1})$, which is the continuously compounded rate of return from the share S from time $t - 1$ and time t. Then we assume that the return process $Y := \{Y_t | t \in \mathcal{T}\}$ follows a first-order Flexible Coefficient Generalized

Autoregressive Conditional Heteroscedastic model with $m = H + 1$ limiting regimes, henceforth, FC-GARCH $(m, 1, 1)$:

$$
\begin{aligned}
Y_t &= \mu_t + h_t^{1/2}\epsilon_t \ , \\
h_t &= G(w_t; \psi) \ .
\end{aligned}
\tag{2.2}
$$

Here $G(w_t; \psi)$ is a nonlinear function of a vector of variables $w_t := (Y_{t-1}, h_{t-1}, s_t)'$, (i.e. "$\prime$" represents the transpose of a matrix, or in particular a vector), defined by:

$$
G(w_t; \psi) := \alpha_0 + \beta_0 h_{t-1} + \lambda_0 Y_{t-1}^2 + \sum_{i=1}^{H} [\alpha_i + \beta_i h_{t-1} + \lambda_i Y_{t-1}^2] f(s_t; \gamma_i, c_i) \ ,
$$

where

1. for each $i = 1, 2, \ldots, H$, the logistic function

$$
f(s_t, \gamma_i, c_i) := \frac{1}{1 + e^{-\gamma_i(s_t - c_i)}} \ ;
$$

2. the vector of parameters

$$
\psi := (\alpha_0, \beta_0, \lambda_0, \alpha_1, \cdots, \alpha_H, \beta_1, \cdots, \beta_H, \lambda_1, \cdots, \lambda_H, \gamma_1, \cdots, \gamma_H, c_1, \cdots, c_H)' \in \mathbb{R}^{3+5H} \ ;
$$

3. for each $i = 1, 2, \cdots, N$, the parameter γ_i is the slope parameter. When $\gamma_i \to \infty$, the function becomes a step function. Here, we consider a simple case that $s_t = Y_{t-1}$.

The class of FC-GARCH models provides the flexibility in incorporating the asymmetric effect of the sign and the size of the previous return Y_{t-1} on the current variance level h_t. It can also capture the heavy-tailedness of return's distribution and the slow decay of the autocorrelation of the squared returns process $\{Y_t^2 | t \in \mathcal{T}\}$, (see He and Terasvirta (1999)). In addition, the FC-GARCH model can capture another important stylized empirical feature of returns data, namely, the Taylor effect, first documented by Taylor (1986). The Taylor effect refers to the strong autocorrelation of absolute daily returns data. This also relates to the long-memory effect of volatility; that is, the decay of the autocorrelations of volatility is too slow to be described by any short memory autoregressive moving average time series models. In the empirical studies by Ding et al. (1993), it has been documented that the realized volatility decays in a hyperparabolic rate.

When $\gamma_i = 0$, or $\alpha_i, \beta_i, \lambda_i = 0$, $i = 1, 2, \cdots, H$, the FC-GARCH model reduces to the GARCH(1,1) model. The FC-GARCH model also nests other important ARCH-type models in the literature. Some examples include the LST-GARCH(1,1) model, the GJR-GARCH(1,1) model,

the VS-GARCH(1,1) model, the ANST-GARCH(1,1) model, the DT-ARCH(1,1) model, the DT-GARCH(1,1) model, and others. For detail, interested readers may refer to Veiga and Medeiros (2008).

§3. The Conditional Esscher Transform

In this section, we recall the method of the conditional Esscher transform to determine a price kernel for option valuation. The method applies to determine a price kernel for a general FC-GARCH model in the next section.

For each $t \in \mathcal{T}$, write F_t for the $\mathcal{P}$-completed, σ-field generated by the share price process up to and including time t and write also $F := \{F_t | t \in \mathcal{T}\}$. We assume that under $\mathcal{P}$,

$$Y_t = \mu_t + \xi_t \ .$$

where ξ_t is an i.i.d innovation process having distribution $D(0, h_t)$ and μ_t and h_t are F_{t-1}-measurable.

We now define the conditional Esscher transform. Let $\{\theta_t | t \in \mathcal{T} \backslash \{0\}\}$ be an F-predictable, real-valued, process on $(\Omega, \mathcal{F}, \mathcal{P})$. Denote, for each $t \in \mathcal{T} \backslash \{0\}$, the moment generating function of Y_t given F_{t-1} under $\mathcal{P}$ evaluated at $z \in \Re$ by $M_Y(t, z)$; that is,

$$M_Y(t, z) := E[e^{zY_t} | F_{t-1}] \ .$$

Here E is expectation under $\mathcal{P}$.

Assume that, for each $t \in \mathcal{T} \backslash \{0\}$ and $z \in \Re$, $M_Y(t, z)$ exists, (i.e. $M_Y(t, z) < \infty$). Consider an F-adapted process $\{\Lambda_t | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with $\Lambda_0 = 1$, $\mathcal{P}$-a.s., defined by:

$$\Lambda_t := \prod_{k=1}^{t} \frac{e^{\theta_k Y_k}}{M_Y(k, \theta_k)} \ , \quad t \in \mathcal{T} \backslash \{0\} \ .$$

Then, it is easy to check that $\{\Lambda_t\}_{t \in \tau}$ is an $(F, \mathcal{P})$-martingale. So, $E[\Lambda_T] = 1$.

Now we define a new probability measure $\mathcal{P}^\theta$ equivalent to $\mathcal{P}$ on F_T by setting

$$\left. \frac{d\mathcal{P}^\theta}{d\mathcal{P}} \right|_{F_T} := \Lambda_T \ . \tag{3.1}$$

We call $\mathcal{P}^\theta$ the conditional Esscher transform associated with θ.

Let $M_Y^\theta(t, z)$ be the moment generating function of the return Y_t given F_{t-1} under the new measure $\mathcal{P}^\theta$. Write $E^\theta[\cdot]$ for expectation under $\mathcal{P}^\theta$. Then, by the Bayes' rule, it is easy to check

that

$$M_Y^\theta(t, z) = \frac{M_Y(t, \theta_t + z)}{M_Y(t, \theta_t)} \ . \tag{3.2}$$

According to the fundamental theorem of asset pricing (see Harrsion and Kreps and Harrsion and Pliska (1981, 1983)), the absence of arbitrage opportunities is "essentially" equivalent to the existence of an equivalent martingale measure under which discounted price processes are martingales. We call the latter a martingale condition.

Now we write $\tilde{S}_t := e^{-rt} S_t$, which is the discounted asset price at time t, for each $t \in \mathcal{T}$. Then in our case, the martingale condition is:

$$\tilde{S}_u = E^\theta[\tilde{S}_t | F_u] \ , \quad \text{for all } u, t \in \mathcal{T} \text{ with } u \leq t \ . \tag{3.3}$$

Here E^θ is expectation under $\mathcal{P}^\theta$.

The following proposition gives the necessary and sufficient condition for the martingale condition. The proof is standard, so we only state the result.

Proposition 3.1: The martingale condition is satisfied if and only if there exists an F-predictable process $\{\theta_t | t \in \mathcal{T} \backslash \{0\}\}$ such that

$$r = \ln M_Y(t, \theta_t + 1) - \ln M_Y(t, \theta_t) \ . \tag{3.4}$$

The existence and uniqueness of the process θ can be established using some standard arguments.

Consider an European-style option with payoff $V(S_T)$ at maturity T. Then, a conditional price of the option at time t given F_t is determined as:

$$V_t = e^{-r(T-t)} E^\theta[V(S_T) | F_t] \ . \tag{3.5}$$

§4. Some Parametric Cases

In this section, we consider some parametric cases of our model in Section 2 when the GARCH innovations have a normal distribution and a shifted gamma distribution. The development in this section follows that of Siu et al. (2004).

§4.1 Normal innovations

Firstly, under $\mathcal{P}$, $Y_t|F_{t-1} \sim N(\mu_t, h_t)$. Consequently,

$$
\begin{aligned}
\ln(M_{Y_t|F_{t-1}}(1, \theta_t)) &= \ln\left(\frac{M_{Y_t|F_{t-1}}(1 + \theta_t)}{M_{Y_t|F_{t-1}}(\theta_t)} \right) \\
&= \ln\left(\frac{e^{\mu_t(1+\theta_t) + \frac{(1+\theta_t)^2 h_t}{2}}}{e^{\mu_t \theta_t \frac{\theta_t^2 h_t}{2}}} \right) \\
&= \mu_t + h_t \theta_t + \frac{h_t}{2} \ .
\end{aligned}
$$

Then, the martingale condition implies that

$$
\theta_t = \frac{r - \mu_t - \frac{h_t}{2}}{h_t} \ .
$$

It is not difficult to see that

$$
\begin{aligned}
M_{Y_t|F_{t-1}}(z, \theta_t) &= \frac{M_{Y_t|F_{t-1}}(z + \theta_t)}{M_{Y_t|F_{t-1}}(\theta_t)} \\
&= e^{z\left(r - \frac{h_t}{2}\right) + \frac{z^2 h_t}{2}} \ .
\end{aligned}
$$

This is the moment generating function of a normal distribution with mean $r - \frac{h_t}{2}$ and variance h_t.

Let $\epsilon_t^\theta := \xi_t - r + \mu_t + \frac{h_t}{2}$, for each $t \in \mathcal{T} \backslash \{0\}$. Then under $\mathcal{P}^\theta$, $\epsilon_t^\theta|F_{t-1} \sim N(0, h_t)$. Further, if we take $\mu_t := r + \lambda\sqrt{h_t} - \frac{1}{2}h_t$ as in Duan (1995),

$$
\epsilon_t^\theta = \xi_t + \lambda\sqrt{h_t} \ .
$$

Consequently, under $\mathcal{P}^\theta$, the conditional variance dynamics are given by:

$$
\begin{aligned}
h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0 (\epsilon_{t-1}^\theta - \lambda\sqrt{h_{t-1}})^2 \\
&+ \sum_{i=1}^{H} \left[\alpha_i + \beta_i h_{t-1} + \lambda_i (\epsilon_{t-1}^\theta - \lambda\sqrt{h_{t-1}})^2 \right] f(s_t; \gamma_i, c_i) \ .
\end{aligned}
$$

§4.2 Shifted-Gamma Innovations

For each $t \in \mathcal{T} \backslash \{0\}$, $X_t \sim Ga(a, b)$, where $Ga(a, b)$ represents a Gamma distribution with shape parameter a and scale parameter b. We now suppose that the innovation at time t is given by:

$$
\xi_t := \sqrt{h_t}\left(\frac{X_t - a/b}{\sqrt{a/b^2}} \right) \ , \tag{4.1}
$$

so we write $\xi_t|F_{t-1} \sim SGa(0, h_t)$.

Then, under $\mathcal{P}$,

$$Y_t \quad = \quad r + \lambda \sqrt{h_t} - \frac{1}{2} h_t + \xi_t \tag{4.2}$$

$$h_t \quad = \quad \alpha_0 + \beta_0 h_{t-1} + \lambda_0 \xi_{t-1}^2 + \sum_{i=1}^{H} (\alpha_i + \beta_i h_{t-1} + \lambda_i \xi_{t-1}^2) f(s_t; \gamma_i, c_i) \ , \tag{4.3}$$

where

$$f(s_t, \gamma_i, c_i) := \frac{1}{1 + e^{-\gamma_i(s_t - c_i)}} \ .$$

The return process Y can be expressed as:

$$Y_t = r + \lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{a h_t} + b \sqrt{\frac{h_t}{a}} X_t \ .$$

Note that $b \sqrt{\frac{h_t}{a}} X_t | F_{t-1} \sim Ga(a, \sqrt{\frac{a}{h_t}})$. Then, $Y_t | F_{t-1}$ is a shifted Gamma random variable with shape parameter a, scale parameter $\sqrt{\frac{a}{h_t}}$ and shift parameter $-r - \lambda \sqrt{h_t} + \frac{1}{2} h_t + \sqrt{a h_t}$. Consequently, the moment generating function of $Y_t | F_{t-1}$ is given by:

$$M_{Y_t | F_{t-1}}(\theta_t) = \left(\frac{\sqrt{\frac{a}{h_t}}}{\sqrt{\frac{a}{h_t}} - \theta_t} \right)^a e^{(r + \lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{a h_t}) \theta_t} \tag{4.4}$$

provided that $\theta_t < \sqrt{\frac{a}{h_t}}$.

Again, using the following formula,

$$M_Y^{\theta_t}(z, \theta_t) = \frac{M_Y(t, \theta_t + z)}{M_Y(t, \theta_t)} \ , \tag{4.5}$$

it is not difficult to show that

$$M_{Y_t | F_{t-1}}(z, \theta_t) \quad = \quad \left(\frac{\sqrt{\frac{a}{h_t}} - \theta_t}{\sqrt{\frac{a}{h_t}} - \theta_t - z} \right)^a e^{(r + \lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{a h_t}) z} \ , \tag{4.6}$$

provided that $z < \sqrt{\frac{a}{h_t}} - \theta_t$.

Consequently, the martingale condition implies that

$$\theta_t = \sqrt{\frac{a}{h_t}} - \left[1 - e^{\frac{\lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{a h_t}}{a}} \right]^{-1} \ . \tag{4.7}$$

Now if we take $b_t := \sqrt{\frac{a}{h_t}}$ and $b_t^\theta := \left[1 - e^{\frac{\lambda \sqrt{h_t} - \frac{1}{2} h_t - \sqrt{a h_t}}{a}} \right]^{-1}$, then

$$b_t^\theta = \theta_t - b_t \ .$$

Under $\mathcal{P}^\theta$,

$$Y_t|F_{t-1} \sim SGa(a, b_t^\theta, -r - \lambda\sqrt{h_t} + \frac{1}{2}h_t + \sqrt{ah_t}) \ .$$

Let $\sim$ be "equal in distribution". Then, we can write

$$Y_t \sim r + \lambda\sqrt{h_t} - \frac{1}{2}h_t - \sqrt{ah_t} + X_t^\theta \ .$$

Here $X_t^\theta \sim Ga(a, b_t^\theta)$, and

$$\begin{aligned} h_t &= \alpha_0 + \beta_0 h_{t-1} + \lambda_0(X_{t-1}^\theta - \sqrt{ah_{t-1}})^2 \\ &+ \sum_{i=1}^{H}[\alpha_i + \beta_i h_{t-1} + \lambda_i(X_{t-1}^\theta - \sqrt{ah_{t-1}})^2]f(s_t; \gamma_i, c_i) \ . \end{aligned}$$

§5. Simulation Studies

In this setion, we start by conducting simulation exercices for the option price with an underlying asset which logreturn follows the FC-GARCH. We perform the simulation in two cases: with innovations having normal and shifted-Gamma distributions. We simulate FC-GARCH prices and use it to estimate a GARCH so that we can compare the FC-GARCH option prices to option prices generated by a GARCH simulated with the estimated parameters. We also compare them to the Black Scholes option price based on a MGB. In the end of the section, we do a sensibility analysis study to check how sensible the option price is according to changes in each of the parameters.

We take an option with maturity $T = 63$, we consider 252 trading days per year and we set 0.1187 as the annual volatility. We perform 10000 simulations for the option prices and use both the antithetic and control variate techniques in the normal case and the control variate in the Gamma case.

The parameters used for simulating the FC-GARCH option prices are:

For the Shifted-Gamma case, we also assume that $a = 0.567$.

In order to compare the FC-GARCH option price with an option price based on a GARCH(1,1), we obtain the parameters to do the GARCH simulation by a iterated two-stage method of estimation, in which we first estimate the risk-premium based on the comparison of the conditional mean found by the MATLAB[1] and the actual equation for the conditional mean. Using this risk-premium,

---

[1]We used the garchfit command in MATLAB.

FC-GARCH Parameters	
α	$[2.22 \times 10^{-16}, 2.55 \times 10^{-5}, 3.73 \times 10^{-4}]$
β	$[1.5186, -0.6339, -0.7238]$
λ	$[0.0438, -0.0113, -0.0286]$
γ	$[551.71, 413.78]$
c	$[-0.0324, 0.0407]$
Risk Premium	0.0359

Table 1: Vectors of parameters for the FC-GARCH including the values of α, β and λ in the three different regimes.

we rescale the data subtracting the conditional mean and estimate the GARCH parameters separatedly. We performed 1000 iterations and the parameters converged. The variance of the data used to do the estimation was 4.0789×10^{-4}. This was used as the initial variance to perform the simulation for finding the option prices.

GARCH Parameters in Normal Case	
α	3.2822×10^{-5}
β	0.8265
λ	0.0928
Risk Premium	0.1221

For the shifted gamma noises, we estimated the a using the two stage procedure as in Siu et al(2004) by using the formula:

$$\hat{a} = \left[\frac{2 \sum_{t=1}^{T} h_t^{3/2}}{\sum_{t=1}^{T} \xi_t^3} \right]^2 \tag{4.8}$$

which led us to the following parameters:

GARCH Parameters in Shifted-Gamma Case	
α	4.2816×10^{-5}
β	0.8814
λ	0.0179
Risk Premium	0.0349
a	0.5114

The variance of the data was 4.2330×10^{-4}. This was used as the initial variance to perform the simulation for finding the option prices.

Tables 1 and 2 displays the option prices obtained from the FC-GARCH model with normal innovations and Tables 3 and 4 displays the option prices obtained from the FC-GARCH model with shifted gamma noises. In the four tables we compare it to the associated Black-Scholes and GARCH prices.

Note that the FC-GARCH always overprice the Black Scholes case and the GARCH model in the normal case. On the other hand, the FC-GARCH always underprice the Black Scholes as well as it underprices the GARCH in the gamma case.

Call Prices for the FC-GARCH with IV=1.0

K/S_0	BS	FCNormal	GARCH
0.80	20.7062	20.8431	20.5368
0.90	12.6986	12.7960	12.2975
0.95	9.4850	9.5540	8.9856
1.00	6.8628	6.9350	6.3299
1.05	4.8140	4.9089	4.3114
1.10	3.2783	3.4187	2.8579
1.20	1.4010	1.6072	1.1569

Call Prices for the FC-GARCH with IV=1.2

K/S_0	BS	FCNormal	GARCH
0.80	20.7062	20.8794	20.6579
0.90	12.6986	12.8682	12.5015
0.95	9.4850	9.6534	9.2326
1.00	6.8628	7.0434	6.5861
1.05	4.8140	5.0230	4.5422
1.10	3.2783	3.5260	3.0505
1.20	1.4010	1.7167	1.2873

Call Prices for the FC-GARCH with IV=1.0

K/S_0	BS	FCSG	GARCH
0.80	20.5751	20.1627	20.3600
0.90	12.4219	11.6073	11.9746
0.95	9.1546	8.2994	8.7076
1.00	6.5076	5.7375	6.1263
1.05	4.4651	3.8441	4.1954
1.10	2.9613	2.4987	2.8028
1.20	1.1855	1.0104	1.2002

Call Prices for the FC-GARCH with IV=1.2

K/S_0	BS	FCSG	GARCH
0.80	20.5751	20.3908	20.5503
0.90	12.4219	11.7655	12.2693
0.95	9.1546	8.3907	9.0418
1.00	6.5076	5.7701	6.4839
1.05	4.4651	3.8468	4.5429
1.10	2.9613	2.5124	3.1006
1.20	1.1855	1.0662	1.3988

§6. Summary and discussion

In this paper we adopted the method of Siu et al(2004) to find the risk neutral version of the FC-GARCH with two different innovations, the normal and the shifted-gamma cases. We also performed simulations and showed tables comparing the Black Scholes price and the GARCH price to our simulation results of the FC-GARCH. We noticed that our prices slightly overprice the Black Scholes price and the GARCH price in the normal case but in the gamma case, the FC-GARCH underprice both the Black Scholes and the GARCH.

REFERENCES

DUAN, Jin-Chuan. 1995. "The GARCH Option Pricing Model," *Mathematical Finance* 5: 13-32.

SIU, Tak Kuen, Howell Tong, and Hailiang Yang. 2004.

MEDEIROS, Marcelo C. and Álvaro Veiga. 2008. "Modeling Multiple Regimes in Financial Volatility with a Flexible Coefficient GARCH Model",*Econometric Theory.*