Dynamic signatures of a coherent system

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Resumo. Neste artigo apresentamos condições sob as quais definimos um processo de assinaturas para um sistema coerente observando o tempo de vida dos componentes, ordenados, como aparecem no tempo, até a falha do sistema. No caso dos tempos de vida dos componentes independentes e identicamente distribuidos, o processo de assinaturas atualiza-se no tempo e não depende da particular função de distribuição, F, dos tempos de vida dos componentes e do instante t. O processo tambem recupera as assinaturas no infinito.

Abstract. In this paper we give conditions,under which, we can define a signature process of a coherent system observing the ordering components lifetimes, as they appear in time until system failure. In the case of independent and identically distributed component lifetimes, the signature process actualizes itself on time and does not depend of the particular components lifetimes distribution F and of the particular time t. Its also recover the system signature at infinity.

Keywords: Dynamic system signature; Coherent systems; Point processes.

1.Introduction. As in Barlow and Proschan (1981) a complex engineering system is completely characterized by its structure function Φ which relate its lifetime T and its components lifetimes T_i, $1 \leq i \leq n$, defined in a complete probability space $(\Omega, \Im, P)$

$$T = \Phi(\mathbf{T}), \mathbf{T} = (T_1, ..., T_n).$$

A physical system would be quit unusual (or perhaps poorly designed) if improving the performance of a component (that is, replacing a failed component by a functioning component) caused the system to deteriorate (that is, to change from the functioning state to the failed state). Thus we consider structure functions which are monotonically increasing in each coordinate. Also to avoid trivialities we will eliminate consideration of any system whose state does not depend on the state of its components. A system is said to be coherent if its structure function Φ is increasing and each component is relevant, that is, there exist a time t and a configuration of $\mathbf{T}$ in t such that the system works if, and only if, the component works.

The performance of a coherent system can be measured from this structural relationship and the distribution function of its components lifetimes. The structure functions offer a way of indexing the class of coherent system but such representations make the distribution function of the system lifetime analytically very complicated (mainly in the dependent case). An alternative representation for the coherent system distribution function is through the system signatures, as in Samaniego (2007), that, while narrower in scope than the structure function, is substantially more useful.

Samaniego (1985) consider the order statistics of the independent and identically distributed components lifetimes of a coherent system of order n with continuous distribution. Clearly $\{T = T_{(i)}\}$ $1 \leq i \leq n$ is a (P-a.s.) partition of the probability space and

$$P(T \leq t) = \sum_{i=1}^{n} P(T \leq t, T = T_{(i)}) = \sum_{i=1}^{n} P(T = T_{(i)}) P(T \leq t | T = T_{(i)}) =$$

$$\sum_{i=1}^{n} P(T = T_{(i)}) P(T_{(i)} \leq t | T = T_{(i)}) = \sum_{i=1}^{n} P(T = T_{(i)}) P(T_{(i)} \leq t) =$$

$$\sum_{i=1}^{n} \alpha_i P(T_{(i)} \leq t).$$

In the above context Samaniego (2007) defines

Definition 1.1 Let T be the lifetime of a coherent system of order n, with components lifetimes $T_1, ..., T_n$ which are independent and identically distributed random variables with continuous distribution F. Then the signature vector α is defined as

$$\alpha = (\alpha_1, ..., \alpha_n)$$

where $\alpha_i = P(T = T_{(i)})$ and the $\{T_{(i)}, 1 \leq i \leq n\}$ are the order statistics of $\{T_i, 1 \leq i \leq n\}$.

The key feature of system signatures that makes them broadly useful in reliability analysis is the fact that, in the context of independent and identically distributed (i.i.d.) absolutely continuous components lifetimes, they are distribution free measures of system quality, depending solely on the design characteristics of the system and independent of the behavior of the systems components .

A detailed treatment of the theory and applications of system signatures may be found in Samaniego (2007). This reference gives detailed justification for the i.i.d. assumption used in the definition of system signatures. By the way there are some applications in which the i.i.d. assumption is appropriate, and in such case, the use of system signatures for comparisons among systems is wholly appropriate; such applications range from batteries in lighting, to wafers or chips in a digital computer to the subsystem of spark plugs in an automobile engine.

The utility of signatures in gauging the performance of systems in i.i.d. components derives largely from representation and preservation results. Some of then link the characteristics of system signatures with system performance.

Before stating these results, we first recall the definitions of three standard forms of stochastic relations between random variables.

Definition 1.2 Let T_1 and T_2 random variables. Then:

a) T_1 is stochastically smaller than T_2 ($T_1 \leq_{st} T_2$) if, and only if, $P(T_1 > t) \leq P(T_2 > t), \forall t$;

b) T_1 is stochastically smaller than T_2 in the hazard rate ordering ($T_1 \leq_{hr} T_2$) if, and only if, $\frac{P(T_1 > t)}{P(T_2 > t)}$ is nonincreasing in $t, \forall t$;

c) in the case where T_1 and T_2 have absolutely continuous distributions, with densities f_1 and f_2, respectively, T_1 is stochastically smaller than T_2 in the likelihood rate ordering ($T_1 \leq_{lr} T_2$) if, and only if, $\frac{f_1(t)}{f_2(t)}$ are nonincreasing in $t, \forall t$.

The following result shows that certain relationships between two (discrete) signatures ensure that a similar relationship holds between the corresponding (continuous) system lifetimes.

Theorem 1.3(Kochar et al. (1999)) Let $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ be the signatures of two coherent systems of order n, both based on n components with i.i.d. lifetimes with common continuous distribution F. Let S_1 and S_2 be their respective lifetimes. Then:

a) if

$$\boldsymbol{\alpha_1} \leq_{st} \boldsymbol{\alpha_2} \Longrightarrow S_1 \leq_{st} S_2;$$

b) if

$$\boldsymbol{\alpha_1} \leq_{hr} \boldsymbol{\alpha_2} \Longrightarrow S_1 \leq_{hr} S_2;$$

c) if F is absolutely continuous and

$$\alpha_{\mathbf{1}} \leq_{lr} \alpha_{\mathbf{2}} \Longrightarrow S_1 \leq_{lr} S_2.$$

The applications of system signature can be extended to mixed system. A mixed system of order n is a stochastic mixture of coherent systems of order n and can be realized in practice via randomization which selects a system at random according to a fixed probability distribution on the class of coherent systems of order n (see Boland et al. (2005)). The mixed system that selects among n-component systems with signatures vectors $\alpha_{\mathbf{1}}, \alpha_{\mathbf{2}}, ..., \alpha_{\mathbf{n}}$ according to the distribution $\mathbf{p} = (p_1, ..., p_n)$ will have signature $\sum_{i=1}^{n} p_i \alpha_{\mathbf{i}}$. We note that the representation and preservation theorem above is applicable for mixed systems.

One further important issue is the fact that we will, at times, be interested in comparing systems of different sizes. Although such a comparison might arise in general, it is special relevant when comparisons involve new and used systems. Theorem 1.3 is not immediately applicable to this problem. However, the exact relationship has been characterized between the signature of a given system with a system of any larger order, which has an equivalent lifetime distribution under the assumption of i.i.d. component lifetimes. The following theorem is an example.

Theorem 1.4 (Samaniego (2006)) Let $\alpha = (\alpha_1, ..., \alpha_n)$ be the signature of a mixed system in n i.i.d. components lifetimes with continuous distribution F. Then the mixed system with $(n+1)$ i.i.d. components lifetimes with continuous distribution F and corresponding to the system signature

$$\alpha^* = (\frac{n\alpha_1}{n+1}, \frac{\alpha_1 + (n-1)\alpha_2}{n+1}, \frac{2\alpha_2 + (n-2)\alpha_3}{n+1}, ..., \frac{(n-1)\alpha_{n-1} + \alpha_n}{n+1}, \frac{n\alpha_n}{n+1})$$

has the same distribution lifetime as the n-component system with signature α.

Samaniego (1985), Kochar, et al.(1999) and Shaked and Suarez-Llorens (2003) extended the signature concept to the case where the components lifetimes $T_1, ..., T_n$, of a system are exchangeable (i.e. the joint distribution function, $F(t_1, ..., t_n)$, of $(T_1, ..., T_n)$ is the same for any permutation of $t_1, ..., t_n$), an interesting and practical situation in reliability theory.

Navarro et al. (2008) and Samaniego et al.(2009) consider dynamic (conditioned) signatures and their use in comparing the reliability of new and used systems. Their procedures consider the system lifetime conditioned in an event on time. Navarro et al. (2008) consider either the event $\{T > t\}$ and $\{T_{(i)} \leq t\} \cap \{T > t\}$ with system signature $P(T = T_{(i)}|T > t)$ and $P(T = T_{(i)}|\{T_{(i)} \leq t\} \cap \{T > t\})$ respectively. A systems signature has proven to be quite a useful proxy for a systems design, as it is a distribution-free measure (i.e., not depending on F) that efficiently captures the precise features of a systems design which influence it performance. Unhappiness, in both Navarros above situations, the system signatures does depend on $F(t)$.

Samaniego et all. (2009) consider the event in time $\{T_{(i)} \leq t < T_{(i+1)}\} \cap \{T > t\}$ and in this case the system signature $P(T = T_{(i)}|\{T_{(i)} \leq t < T_{(i+1)}\} \cap \{T > t\})$ does not depend on t and on $F(t)$ and have the usual signatures properties. Samaniego et all. (2009) extend Theorem 1.3, however their conditioned signature does not capture the dynamics aspects of the problem, as a stochastic process.

Theorem 1.5 (Samaniego et al (2009)) Consider two used mixed systems with lifetimes S_1 and S_2, based on n original components with i.i.d. lifetimes having the common continuous distribution F. Suppose both systems are working and have exactly i and j failed component, respectively, at time t. Let $\alpha_{\mathbf{1}}(\mathbf{n}-\mathbf{i})$ and $\alpha_{\mathbf{2}}(\mathbf{n}-\mathbf{j})$ be their dynamics signatures, as in Samaniego, et al. (2009)[11]. Then

a) if

$$\alpha_{\mathbf{1}}(\mathbf{n}-\mathbf{i}) \leq_{st} \alpha_{\mathbf{2}}(\mathbf{n}-\mathbf{j}) \Longrightarrow$$

$$(S_1|\{T_{(i)} \leq t < T_{(i+1)}\} \cap \{S_1 > t\}) \leq_{st} (S_2|\{T_{(j+1)} \leq t < T_{(j+2)}\} \cap \{S_2 > t\});$$

b) if

$$\alpha_{\mathbf{1}}(\mathbf{n}-\mathbf{i}) \leq_{hr} \alpha_{\mathbf{2}}(\mathbf{n}-\mathbf{j}) \Longrightarrow$$

$$(S_1|\{T_{(i)} \leq t < T_{(i+1)}\} \cap \{S_1 > t\}) \leq_{hr} (S_2|\{T_{(j+1)} \leq t < T_{(j+2)}\} \cap \{S_2 > t\});$$

c) if F is absolutely continuous and if

$$\alpha_{\mathbf{1}}(\mathbf{n}-\mathbf{i}) \leq_{lr} \alpha_{\mathbf{2}}(\mathbf{n}-\mathbf{j}) \Longrightarrow$$

$$(S_1|\{T_{(i)} \leq t < T_{(i+1)}\} \cap \{S_1 > t\}) \leq_{lr} (S_2|\{T_{(j+1)} \leq t < T_{(j+2)}\} \cap \{S_2 > t\});$$

In this paper we consider the system evolution on time under a complete information level. To make the exposition understandable, in Section 1, in the introduction, we review results in signature theory and in Section 2 we develop the dynamics signature. We give some examples at the end of this Section.

2. Dynamic signatures

We intend to give a new approach to dynamic systems signatures. We consider the system evolution on time under a complete information level. In this fashion, if the components lifetimes are independent and identically distributed and continuous, the expected dynamic system signature enjoy the special property that they are independent of both the distribution F and the time t. This fact has significance beyond the mere simplicity and tractability of the signature vector, reflect only characteristics of the corresponding system design and may be used as proxies for system designs in the comparison of system performance. Also the dynamic system signature actualizes itself under the system evolution on time recovering the dynamical system signature in the set $\{T_{(i)} \leq t < T_{(i+1)}\} \cap \{T > t\}$, as in Samaniego et all (2009) and the original coherent system signature in the set $\{T_{(n)} \leq t\}$ as in Samaniego (2007).

In our setup, we consider the vector $(T_1, ..., T_n)$ of n component lifetimes which are finite, positive, independent and identically distributed and continuous random variables defined in a complete probability space $(\Omega, \Im, P)$. (in a general setup we can assume that $P(T_i \neq T_j) = 1$, for all $i \neq j, i, j$ in $E = \{1, ..., n\}$, the index set of components. The lifetimes can be dependent but simultaneous failure are ruled out).

In what follows, to simplify the notation, we assume that relations between random variables and measurable sets, respectively, always hold with probability one, which means that the term P-a.s., is suppressed.

The evolution of components in time define a marked point process given through the failure times and the corresponding marks.

We denote by $T_{(1)} < T_{(2)} < ... < T_{(n)}$ the ordered lifetimes $T_1, T_2, ..., T_n$, as they appear in time and by $X_i = \{j : T_{(i)} = T_j\}$ the corresponding marks. As a convention we set $T_{(n+1)} = T_{(n+2)} = ... = \infty$ and $X_{n+1} = X_{n+2} = ... = e$ where e is a fictitious mark not in E. Therefore the sequence $(T_n, X_n)_{n \geq 1}$ defines a marked point process.

The mathematical formulation of our observations is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t \geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(i)} > s\}}, X_i = j, 1 \leq j \leq n, j \in E, 0 < s \leq t\},$$

satisfies the Dellacherie conditions of right continuity and completeness .

Intuitively, at each time t the observer knows if the events $\{T_{(i)} \leq t, X_i = j\}$ have either occurred or not and if they have, he knows exactly the value $T_{(i)}$ and the mark X_i. We assumed that $T_i, 1 \leq i \leq n$ are totally inaccessible $\Im_t$-stopping time. To prove the main result we need the following Lemma:

Lemma 2.1.1 Let T be the lifetime of a coherent system of order n, with component lifetimes $T_1, ..., T_n$ which are independent and identically distributed with continuous distribution F. Then, under the above notation

$$P(T \leq t | \Im_t) = \Sigma_{i=1}^{n} \frac{P(T = T_{(i-1)})}{P(T \geq T_{(i-1)})} 1_{\{T_{(i-1)} \leq t < T_{(i)}\}}.$$

Proof

Under the assumption that the components lifetimes are i.i.d. with continuous distribution F, $P(\cup_{i=1}^{n} \{T = T_{(i)}\}) = 1$. The information in $\Im_t$ is of the kind $\{T_{(i)} \leq t < T_{(i+1)}\}$, $i = 1, 2, ..., n$ with $T_{(0)} = 0$ and $T_{(n+1)} = \infty$. Therefore a version of $P(T \leq t | \Im_t)$ is

$P(T \leq t | \Im_t) = 0$ in the set $\{T_{(1)} > t\}$;

$P(T \leq t | \Im_t) = \frac{P(T = T_{(i)})}{P(T \geq T_{(i)})}$ in the set $\{T_{(i)} \leq t < T_{(i+1)}\}$;

$P(T \leq t | \Im_t) = 1$ in the set $\{t \geq T_{(n)}\}$.

which can be easily verified observing that

$$\int_A P(T \leq t|\Im_t)dP = P(A \cap \{T \leq t\})$$

$\forall A \in \Im_t$ and we have

$$P(T \leq t|\Im_t) = \Sigma_{i=1}^n \frac{P(T = T_{(i-1)})}{P(T \geq T_{(i-1)})} 1_{\{T_{(i-1)} \leq t < T_{(i)}\}}.$$

To prove the main result we need a definition from Arjas and Norros (1984) and Norros (1985).

Definition 2.1.2 A random vector $\mathbf{T} = (T_1, ..., T_n)$ of lifetimes is said to be weakened by failures with respect to a family of σ-algebra $(\Im_t)_{t \geq o}$, denoted by $WBF|\Im_t$ if for all increasing and measurable function f, $E[f(\mathbf{T})|\Im_t]$ jumps downwards at the failure times.

Remark 2.1.3 Follows that if $f(t) = 1_{\{T>t\}}$, and $\Im_t = \Re_t$ we have $E[f(\mathbf{T})|\Im_t] = P(T > t|\Im_t)$. As, for $T_{(i-1)} \leq t < T_{(i)}$, $\Im_t - = \Im_{T_{(i-1)}}$ we conclude that

$$P(T > T_{(i)}|\Im_{T_{(i)}}) \leq P(T > T_{(i-1)}|\Im_{T_{(i-1)}})$$

which is equivalent to

$$\frac{P(T = T_{(i)})}{P(T \geq T_{(i)})} \geq \frac{P(T = T_{(i-1)})}{P(T \geq T_{(i-1)})}$$

and $\beta_i \geq 0$ under the $WBF|\Im_t$ property of the components lifetimes.

Theorem 2.1.4 Let T be the lifetime of a coherent system of order n, with component lifetimes $T_1, ..., T_n$ which are independent and identically distributed with continuous distribution F and with the $WBF|\Im_t$ property. Then,

$$P(T \leq t|\Im_t) = \Sigma_{i=1}^n \beta_i 1_{\{T_{(i)} \leq t\}}$$

where

$$\beta_i = \frac{P(T = T_{(i)})}{P(T \geq T_{(i)})} - \frac{P(T = T_{(i-1)})}{P(T \geq T_{(i-1)})},$$

with $T_{(0)} = 0$, $T_{(n+1)} = \infty$, $\beta_i \geq 0$ and $\Sigma_{i=1}^n \beta_i = 1$.

Proof Firstly, under the $WBF|\Im_t$ property the β_i are clearly positives.

Also

$$\Sigma_{i=1}^n \beta_i = \Sigma_{i=1}^n [\frac{P(T = T_{(i)})}{P(T \geq T_{(i)})} - \frac{P(T = T_{(i-1)})}{P(T \geq T_{(i-1)})}] = \frac{P(T = T_{(n)})}{P(T \geq T_{(n)})} = 1$$

if $P(T = T_{(n)}) \neq 0$ and equal to $\frac{P(T=T_{(n-1)})}{P(T \geq T_{(n-1)})} = 1$, if $P(T = T_{(n)}) = 0$ and so successively.

From Remark 2.1.1, we have

$$P(T \leq t|\Im_t) = \Sigma_{i=1}^n \frac{P(T = T_{(i-1)})}{P(T \geq T_{(i-1)})} 1_{\{T_{(i-1)} \leq t < T_{(i)}\}} =$$

$$\Sigma_{i=1}^n \frac{P(T = T_{(i-1)})}{P(T \geq T_{(i-1)})} [1_{\{T_{(i)} > t\}} - 1_{\{T_{(i-1)} > t\}}] =$$

$$1 - \Sigma_{i=1}^n [\frac{P(T = T_{(i)})}{P(T \geq T_{(i)})} - \frac{P(T = T_{(i-1)})}{P(T \geq T_{(i-1)})}] 1_{\{T_{(i)} > t\}} =$$

$$\Sigma_{i=1}^n \beta_i - \Sigma_{i=1}^n \beta_i 1_{\{T_{(i)} > t\}} = \Sigma_{i=1}^n \beta_i 1_{\{T_{(i)} \leq t\}}.$$

Therefore, under the $WBF|\Im_t$ property of the components lifetimes the β_is define a probability distribution and we can define

Definition 2.1.5 Let T be the lifetime of a coherent system of order n, with component lifetimes $T_1, ..., T_n$ which are weakened by failure relative to $(\Im_t)_{t \geq 0}$, $WBF|\Im_t$, independent and identically distributed random variables with continuous distribution F. Then the dynamic signature vector β is defined as

$$\beta = (\beta_1, ..., \beta_n)$$

where $\beta_i = \frac{P(T=T_{(i)})}{P(T \geq T_{(i)})} - \frac{P(T=T_{(i-1)})}{P(T \geq T_{(i-1)})}$, and the $T_{(i)}$ are the order statistics of $T_i, 1 \leq i \leq n$.

Remarks 2.1.6: Given the information $\Im_t$ we knows that, in the set $\{T_{(i)} \leq t < T_{i+1)}\}$,

$$P(T \leq t|\{T_{(i)} \leq t < T_{i+1)}\}) = \sum_{j=1}^{i} \beta_i = \sum_{j=1}^{i} [\frac{P(T = T_{(i)})}{P(T \geq T_{(i)})} - \frac{P(T = T_{(i-1)})}{P(T \geq T_{(i-1)})}] = \frac{P(T = T_{(i)})}{P(T \geq T_{(i)})},$$

as we have in Lemma 2.1.1.

The next Corollary shows how the dynamic signature actualizes itself on time and how we recover the Samaniego (2007) signature vector at infinity.

We use the notation from Samaniego (2009): $\beta_{n-i,j}(n-i)$ is the j-th element of the signature vector $\beta_{n-i}(n-i)$ of the remaining $n-i$ residual survival lifetimes of a coherent system of order n after its i-th components failures, $1 \leq i \leq n, //i + 1 \leq j \leq n$. The distribution conditional of the system lifetime given that the system is still working and exactly i components have failed is denoted by

$$P(T \leq t + x|\{T_{(i)} \leq t < T_{i+1)}\} \cap \{T > t\}).$$

Corollary 2.1.7 Let T be the lifetime of a coherent system of order n, with component lifetimes $T_1, ..., T_n$ which are $WBF|\Im_t$, independent and identically distributed with continuous distribution F. Then, in the set $\{T_{(i)} \leq t < T_{(i+1)}\} \cap \{T > t\}$, the system signature actualizes in time with

$$P(T \leq t + x | \{T_{(i)} \leq t < T_{i+1)}\} \cap \{T > t\}) =$$

$$\sum_{j=i+1}^{n} \beta_{n-i,j}(n-i) P(T_{(j)} \leq t + x | \{T_{(i)} \leq t < T_{i+1)}\} \cap \{T > t\}),$$

and $\sum_{j=i+1}^{n} \beta_{n-i,j}(n-i) = 1$. Also, its restores the Samaniego (2007) system signature in the set $\{t \geq T_{(n)}\}$.

Proof In the set $\{T > t\} \cap \{T_{(i)} \leq t < T_{(i+1)}\}$, we have $1_{\{T_{(j)} \leq t\}} = 0$ if $j \leq i$. Also, under this information $P(T = T_{(j)}) = 0$ if $j \leq i$ and $P(T \geq T_{(i+1)}) = 1$ affecting all the differences defining the dynamics signature, passing from β_j to $\beta_{n-i,j}(n-i)$.

Therefore we have

$$P(T \leq t + x | \Re_t) = \sum_{j=i+1}^{n} \beta_{n-i,j}(n-i) 1_{\{T_{(j)} \leq t + x\}},$$

and the coherent system signature actualizes to

$$P(T \leq t + x | \{T_{(i)} \leq t < T_{i+1)}\} \cap \{T > t\}) =$$

$$\sum_{j=i+1}^{n} \beta_{n-i,j}(n-i) P(T_{(j)} \leq t + x | \{T_{(i)} \leq t < T_{i+1)}\} \cap \{T > t\}),$$

with

$$\sum_{j=i+1}^{n} \beta_{n-i,j}(n-i) = \sum_{j=i+1}^{n} [\frac{P(T = T_{(j)})}{P(T \geq T_{(j)})} - \frac{P(T = T_{(j-1)})}{P(T \geq T_{(j-1)})}] = \frac{P(T = T_{(n)})}{P(T \geq T_{(n)})} = 1,$$

if $P(T = T_{(n)}) \neq 0$ and equal to $\frac{P(T = T_{(n-1)})}{P(T \geq T_{(n-1)})} = 1$, if $P(T = T_{(n)}) = 0$ and so successively.

Furthermore, under this information, as $P(T = T_{(i)}) = 0$ and $P(T \geq T_{(i+1)}) = 1$ we have

$$\beta_{n-i,i+1}(n-i) = \frac{P(T = T_{(i+1)})}{P(T \geq T_{(i+1)})} - \frac{P(T = T_{(i)})}{P(T \geq T_{(i)})} = \frac{P(T = T_{(i+1)})}{P(T \geq T_{(i+1)})} = P(T = T_{(i+1)}) = \alpha_{i+1}$$

and the signatures actualizes itself in the set $\{T > t\} \cap \{T_{(i)} \leq t < T_{(i+1)}\}$.

As $\beta_1 = \alpha_1$ and $\{T_i \leq t\}$ occurs successively in time for $i = 1, 2, 3, ...,$ in the set $\{t \geq T_{(n)}\}$ we have:

$$P(T \leq t|\Re_t) = \sum_{i=1}^{n} \alpha_i 1_{\{T_{(i)} \leq t\}}.$$

Taking expected values we get

$$P(T \leq t) = \sum_{i=1}^{n} \alpha_i P(T_{(i)} \leq t)$$

recovering the Samaniego (2007) signature decomposition.

Examples 2.1.8 i) If T_1, T_2, T_3 are independent and identically distributed component's lifetimes of the system with lifetime $T = T_1 \wedge (T_2 \vee T_3)$.

The Samaniego (2007) system signatures are:

$\alpha_1 = P(T = T_{(1)}) = \frac{1}{3}$, $\alpha_2 = P(T = T_{(2)}) = \frac{2}{3}$ and $\alpha_3 = P(T = T_{(3)}) = 0$ and the signature system distribution lifetime decomposition is

$$P(T \leq t) = \frac{1}{3} P(T_{(1)} \leq t) + \frac{2}{3} P(T_{(2)} \leq t).$$

However $\frac{P(T=T_{(1)})}{P(T \geq T_{(1)})} = \frac{1}{3}$, $\frac{P(T=T_{(2)})}{P(T \geq T_{(2)})} = 1$ and $\frac{P(T=T_{(3)})}{P(T \geq T_{(3)})} = 0$ and the dynamical signature are

$\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{2}{3}$, $\beta_3 = 0$ and the dynamic signature system distribution lifetime decomposition is

$$P(T \leq t|\Im_t) = \frac{1}{3} 1_{\{T_{(1)} \leq t\}} + \frac{2}{3} 1_{\{T_{(2)} \leq t\}}.$$

Taking expected values we get

$$P(T \leq t) = \frac{1}{3} P(T_{(1)} \leq t) + \frac{2}{3} P(T_{(2)} \leq t).$$

and note that $\alpha_i = \beta_i, \quad i = 1, 2$ in the set $\{T_{(2)} < t\}$ recovering the Samaniego (2007) signature decomposition.

ii)The Bridge system lifetime can be set as $T = (T_1 \vee T_2) \wedge (T_1 \vee T_3 \vee T_5) \wedge (T_2 \vee T_3 \vee T_4) \wedge (T_4 \vee T_5)$. where T_1, T_2, T_3, T_4, T_5 are independent and identically distributed lifetimes.

The Samaniego (2009) system signatures are:

$\alpha_1 = P(T = T_{(1)}) = 0$, $\alpha_2 = P(T = T_{(2)}) = \frac{1}{5}$, $\alpha_3 = P(T = T_{(3)}) = \frac{3}{5}$, $\alpha_4 = P(T = T_{(4)}) = \frac{1}{5}$ and $\alpha_5 = P(T = T_{(5)}) = 0$ and the signature system distribution lifetime decomposition is

$$P(T \leq t) = \frac{1}{5} P(T_{(2)} \leq t) + \frac{3}{5} P(T_{(3)} \leq t) + \frac{1}{5} P(T_{(4)} \leq t).$$

As $\frac{P(T=T_{(1)})}{P(T\geq T_{(1)})}=0$, $\frac{P(T=T_{(2)})}{P(T\geq T_{(2)})}=\frac{1}{5}$, $\frac{P(T=T_{(3)})}{P(T\geq T_{(3)})}=\frac{3}{4}$, $\frac{P(T=T_{(4)})}{P(T\geq T_{(4)})}=1$ and $\frac{P(T=T_{(5)})}{P(T\geq T_{(5)})}=0$ and follows that the

the dynamical signature are:

$\beta_1=0$, $\beta_2=\frac{1}{5}$, $\beta_3=\frac{11}{20}$, $\beta_4=\frac{1}{4}$,$\beta_5=0$ and the dynamic signature system distribution lifetime decomposition is

$$P(T\leq t|\Im_t)=\frac{1}{5}1_{\{T_{(2)}\leq t\}}+\frac{11}{20}1_{\{T_{(3)}\leq t\}}+\frac{1}{4}1_{\{T_{(4)}\leq t\}}.$$

Now, in the set $\{T_{(2)}\leq t<T_{(3)}\}\cap\{T>t\}$ the system signature actualizes to to $\beta_{5-2,3}(5-2)=\frac{3}{4}$, $\beta_{5-2,4}(5-2)=1-\frac{3}{4}=\frac{1}{4}$ and $\beta_{5-2,5}(5-2)=0$. The decomposition system signature actualizes to

$$P(T\leq t+x|\{T_{(2)}\leq t<T_{(3)}\}\cap\{T>t\})=$$

$$\frac{3}{4}P(T_{(3)}\leq t+x|\{T_{(2)}\leq t<T_{(3)}\}\cap\{T>t\})+\frac{1}{4}P(T_{(4)}\leq t+x|\{T_{(2)}\leq t<T_{(3)}\}\cap\{T>t\}).$$

Also

$$P(T\leq t)=\frac{1}{5}P(T_{(2)}\leq t)+\frac{4}{5}[\frac{3}{4}P(T_{(3)}\leq t)+\frac{1}{4}P(T_{(4)}\leq t)]$$

in the set $\{T_{(4)}\leq t\}$, recovering the Samaniego (2007) signature decomposition.

iii) If T_1,T_2,T_3,T_4 are independent and identically distributed component's lifetimes of the system with lifetime $T=T_1\vee(T_2\wedge T_3\wedge T_4)$, then $\alpha_1=P(T=T_{(1)})=0$, $\alpha_2=P(T=T_{(2)})=\frac{1}{2}$ and $\alpha_3=P(T=T_{(3)})=\frac{1}{4}$ and $\alpha_4=P(T=T_{(4)})=\frac{1}{4}$ and the signature system distribution lifetime decomposition is

$$P(T\leq t)=\frac{1}{2}P(T_{(2)}\leq t)+\frac{1}{4}P(T_{(3)}\leq t)+\frac{1}{4}P(T_{(4)}\leq t).$$

However $\frac{P(T=T_{(1)})}{P(T\geq T_{(1)})}=0$, $\frac{P(T=T_{(2)})}{P(T\geq T_{(2)})}=\frac{1}{2}$, $\frac{P(T=T_{(3)})}{P(T\geq T_{(3)})}=\frac{1}{2}$ and $\frac{P(T=T_{(4)})}{P(T\geq T_{(4)})}=1$.

Follows that the dynamical signature are: $\beta_1=0$, $\beta_2=\frac{1}{2}$, $\beta_3=0$, and $\beta_4=\frac{1}{2}$ the dynamic signature system distribution lifetime decomposition is

$$P(T\leq t|\Im_t)=\frac{1}{2}1_{\{T_{(2)}\leq t\}}+\frac{1}{2}1_{\{T_{(4)}\leq t\}}.$$

In the set $\{T_{(2)}\leq t<T_{(3)}\}\cap\{T>t\}$ we have $\beta_{4-2,3}(4-2)=\frac{1}{2}$ and $\beta_{4-2,4}(4-2)=1-\frac{1}{2}=\frac{1}{2}$. the dynamical system signature actualizes to

$$P(T\leq t+x|\{T_{(2)}\leq t<T_{(3)}\}\cap\{T>t\}))=\frac{1}{2}P(T_{(2)}\leq t+x)+\frac{1}{2}.$$

$$[\frac{1}{2}P(T_{(3)}\leq t+x|\{T_{(2)}\leq t<T_{(3)}\}\cap\{T>t\})+\frac{1}{2}P(T_{(4)}\leq t+x|\{T_{(2)}\leq t<T_{(3)}\}\cap\{T>t\})]$$

$$P(T \leq t) = \frac{1}{2}P(T_{(2)} \leq t) + \frac{1}{2}[\frac{1}{2}P(T_{(3)} \leq t) + \frac{1}{2}P(T_{(4)} \leq t)],$$

recovering the Samaniego (2009) signature decomposition.

Remarks 2.1.9: As the dynamic signatures actualizes itself on time and in the set $\{T_{(n)} < t\}$, $\alpha_i = \beta_i, \ \ \forall i.$ we can rewrite either Theorem 1.3 (Kochar et al. (1999))) and Theorem 1.5 (Samaniego et al. (2009)) in the cited papers in an unified Theorem:

Theorem 2.1.10 Consider two mixed systems based on n original components with i.i.d. lifetimes having the common continuous distribution F and the $WBF|\Im_t$ property. The first system having lifetime S_1, signature vector α^1 and dynamic signature vector β^1. The second one having lifetime S_2, signature vector α^2 and dynamic signature vector β^2. Suppose that at time t, both system are working, the first system have exactly i failed components and the second exactly j failed components. Then

a) if

$$\alpha^{\mathbf{1}}_{\mathbf{n}}(\mathbf{n}-\mathbf{i}) \leq_{st} \alpha^{\mathbf{2}}_{\mathbf{n}}(\mathbf{n}-\mathbf{j}) \Longrightarrow (S_1|\Re_t \leq_{st} (S_2|\Re_t);$$

b) if

$$\alpha^{\mathbf{1}}_{\mathbf{n}}(\mathbf{n}-\mathbf{i}) \leq_{st} \alpha^{\mathbf{2}}_{\mathbf{n}}(\mathbf{n}-\mathbf{j}) \Longrightarrow (S_1|\Re_t) \leq_{hr} (S_2|\Re_t);$$

c) if F is absolutely continuous,

$$\alpha^{\mathbf{1}}_{\mathbf{n}}(\mathbf{n}-\mathbf{i}) \leq_{lr} \alpha^{\mathbf{2}}_{\mathbf{n}}(\mathbf{n}-\mathbf{j}) \ \ \mathit{and} \ \ \beta^{\mathbf{1}}_{\mathbf{n}}(\mathbf{n}-\mathbf{i}) \leq_{lr} \beta^{\mathbf{2}}_{\mathbf{n}}(\mathbf{n}-\mathbf{j}) \Longrightarrow (S_1|\Re_t) \leq_{lr} (S_2|\Re_t);$$

Proof First, we always have $\Sigma_{j=i}^{n}\beta_{n-i,j}(n-i) = 1, \forall i, 1 \leq i \leq n$ in the way that both the vectors β^1 and β^2 are not relevant for stochastically comparing systems lifetimes with respect to stochastic ordering and hazard rate ordering. However, as the stochastic comparisons must holds for all time $t \geq 0$, the constants $\beta_{n-i+1,i}(n-i+1)$ actualizes itself to α_i in the set $\{T_{(i-1)} \leq t < T_{(i)}\} \cap \{T > t\}$ and $\beta_i = \alpha_i \ \ \forall i$ in $t > T_{(n)}$, either the vectors α^1 and α^2 are relevant for those comparisons and are the sufficient conditions.

As the atoms in $\Re_t$ is of the kind $\{T_{(i)} \leq t < T_{(i+1)}\}$, $i = 1, 2, ..., n$ with $T_{(0)} = 0$ and $T_{(n+1)} = \infty$, the proof of parts a) and b) follows from Theorems 1.3 of Kochar et al.(2005) and Theorem 1.5. of Samaniego et al (2009).

To prove part c) we have to consider the likelihood ratio ordering between the vectors β^1 and β^2 and the prove follows as in Theorem 1.5 of Samaniego et al (2009).

Acknowledgements

This work was partially supported by FAPESP, Proc. No. 2010/52227-0.

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