The k-th Chromatic Number of Webs and Antiwebs

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Abstract

A k-fold x-coloring of a graph is an assignment of k distinct colors from the set $\{1,2,\ldots,x\}$ to each vertex such that any two adjacent vertices are assigned disjoint sets of colors. The smallest number x such that G admits a k-fold x-coloring is the k-th chromatic number of G, denoted by $\chi_k(G)$. We determine the exact value of this parameter for webs and antiwebs. As a consequence, we obtain the fractional chromatic number for these graphs. Our results generalize the known corresponding results for odd cycles and imply necessary and sufficient conditions under which $\chi_k(G)$ attains its lower and upper bounds based on the clique and the chromatic numbers. Addionally, we extend the concept of χ-critical graphs to χ_k-critical graphs, and we identify webs and antiwebs having this property.
Keywords: graph coloring, chromatic number, web graph.
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1 Introduction

A *k-fold x-coloring* of a graph is an assignment of k distinct colors to each vertex from the set $\{1,2,\ldots,x\}$ such that any two adjacent vertices are assigned disjoint sets of colors [Ren and Bu (2010); Stahl (1976)]. We say that a graph G is *k-fold x-colorable* if G admits a k-fold x-coloring. For any $k \geq 1$, the smallest number x such that a graph G is k-fold x-colorable is called *k-th chromatic number of* G and is denoted by $\chi_k(G)$ [Stahl (1976)]. Obviously, $\chi_1(G) = \chi(G)$ is the conventional *chromatic number of* G.

This variant of the conventional graph coloring was introduced in the context of radio frequency assignment problem [Narayanan (2002); Opsut and Roberts (1981); Roberts (1979)]. Other applications include scheduling problems, fleet maintenance and traffic phasing problems [Halldórsson and Kortsarz (2002); Opsut and Roberts (1981)].

In this paper, we derive a closed formula for the k-th chromatic number of webs and antiwebs. Our result generalizes that one obtained by Stahl for specific webs, namely odd cycles [Stahl (1976)]. Web and antiwebs form a class of graphs that play an important role in the context of stable sets and vertex coloring [Cheng and Vries (2002a, 2002b); Holm, Torres, and Wagler (2010); Palubeckis (2010); Pêcher and Wagler (2006); Wagler (2004)].

Based on the obtained expression for the k-th chromatic number of web and antiwebs, we derive other results. First, we find necessary and sufficient conditions under which $\chi_k(G)$ attains its lower and upper bounds given by the clique and the chromatic numbers, respectively. Second, we determine the *fractional chromatic number*, which is defined as the minimum ratio $\frac{x}{k}$ among the k-fold x-colorings [Scheinerman and Ullman (1997)]. Addionally, we extend the concept of χ-critical graphs to χ_k-critical graphs, and we identify webs and antiwebs having this property.

Throughout this paper, we mostly use notation and definitions consistent with what is generally accepted in graph theory. Even though, let us set the grounds for all the notation used from here on. Given a graph G, $V(G)$ and $E(G)$ stand for its set of vertices and edges, respectively. The simplified notation V and E is prefered when the graph G is clear by the context. The complement of G is written as $\overline{G} = (V, \overline{E})$. The edge defined by vertices u and v is denoted by uv.

A set $S \subseteq V$ is said to be a *stable* set if the vertices in it are pairwise non-adjacent in G, i.e. $uv \notin E$ $\forall u, v \in S$. The *stability number* $\alpha(G)$ of G is the size of the largest stable set of G. Conversely, a *clique* of G is a subset $K \subseteq V$ of pairwise adjacent vertices. The *clique number* of G is the size of the largest clique and is denoted by $\omega(G)$. A graph G is *perfect* if $\omega(H) = \chi(H)$, for all induced subgraph H of G. The *fractional chromatic number of* G is denoted $\bar{\chi}(G)$. It is well-known that $\omega(G) \leq \bar{\chi}(G) \leq \chi(G)$.

A *chordless cycle* of length n is a graph G such that $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_i v_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_1 v_n\}$. A *hole* is a chordless cycle of length at least four. An *antihole* is the complement of a hole. Holes and antiholes are odd or even according to the parity of their number of vertices. Odd holes and odd antiholes are the minimal imperfect graphs [Chudnovsky, Robertson, Seymour, and Thomas (2006)].

In the next section, we present general lower and upper bounds for the k-th chromatic number of an arbitrary simple graph. The exact value of this parameter is calculated for webs (Subsection 3.1) and antiwebs (Subsection 3.2). Some consequences of this result are also presented. The fractional chromatic number of these graphs are determined in Subsection 3.3. In Subsection 3.4, we identify which webs and antiwebs achieve the bounds given in Section 2. The definitions of χ_k-critical and χ_*-critical graphs are introduced in Section 4, as a natural extension of the concept of χ-critical graphs. Then, we identify some webs as well as all antiwebs that have these two properties.

2 Lower and upper bounds for the k-th chromatic number

Two simple observations lead to lower and upper bounds for the k-th chromatic number of a graph G. On the one hand, every vertex of a clique of G must receive k colors different from any color assigned to the other vertices of the clique. On the other hand, a k-fold coloring can be obtained by just replicating an 1-fold coloring k times. Therefore, we get the following bounds which are tight, for instance, for perfect graphs.

Lemma 1 *For every $k \in \mathbb{N}$, $\omega(G) \leq \bar{\chi}(G) \leq \frac{\chi_k(G)}{k} \leq \chi(G)$.*

Another lower bound is related to the stability number, as follows. The lexicographic product of a graph G by a graph H is the graph that we obtain by replacing each vertex of G by a copy of H and adding all edges between two copies of H if and only if the two replaced vertices of G were adjacent. More formally, the *lexicographic product* $G \bullet H$ is a graph such that:

1. the vertex set of $G \bullet H$ is the cartesian product $V(G) \times V(H)$; and
2. any two vertices $(u, \hat{u})$ and $(v, \hat{v})$ are adjacent in $G \bullet H$ if and only if either u is adjacent to v, or $u = v$ and $\hat{u}$ is adjacent to $\hat{v}$

As noted by Stahl, another way to interpret the k-th chromatic number of a graph G is in terms of $\chi(G \bullet K_k)$, where K_k is a clique with k vertices [Stahl (1976)]. It is easy to see that a k-fold x-coloring of G is equivalent to a 1-fold coloring of $G \bullet K_k$ with x colors. Therefore, $\chi_k(G) = \chi(G \bullet K_k)$. Using this equation we can trivially derive the following lower bound for the k-th chromatic number of any graph.

Lemma 2 *For every graph G and every $k \in \mathbb{N}$, $\chi_k(G) \geq \left\lceil \frac{kn}{\alpha(G)} \right\rceil$.*

Proof: If H_1 and H_2 are two graphs, then $\alpha(H_1 \bullet H_2) = \alpha(H_1)\alpha(H_2)$ [Geller and Stahl (1975)]. Therefore, $\alpha(G \bullet K_k) = \alpha(G)\alpha(K_k) = \alpha(G)$. We get $\chi_k(G) = \chi(G \bullet K_k) \geq \left\lceil \frac{kn}{\alpha(G \bullet K_k)} \right\rceil = \left\lceil \frac{kn}{\alpha(G)} \right\rceil$. □

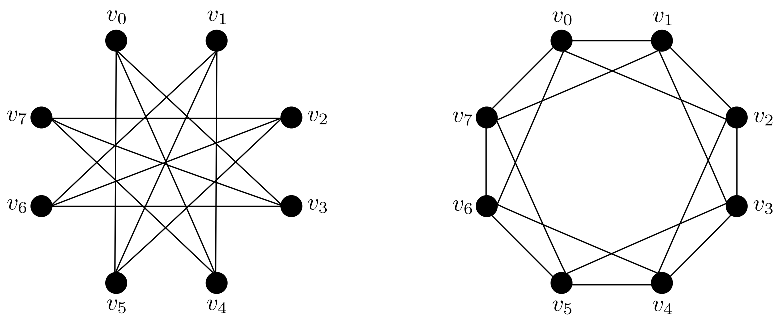
Next we will show that the lower bound given by Lemma 2 is tight for two classes of graphs, namely webs and antiwebs. Moreover, some graphs in these classes also achieve the lower and upper bounds stated by Lemma 1.

3 The k-th chromatic number of webs e antiwebs

Let n and p be integers such that $p \geq 1$ e $n \geq 2p$. The web W_p^n is the graph whose vertices can be labelled as $V(W_p^n) = \{v_0, v_1, \ldots, v_{n-1}\}$ in such a way that $E(W_p^n) = \{(v_i, v_j) \mid v_i, v_j \in V \text{ e } p \leq |i-j| \leq n-p\}$. The antiweb $\overline{W}_p^n$ is defined as the complement of W_p^n. Examples are depicted in Figure 1. Observe that the webs W_1^n are the cliques whereas W_l^{2l+1} and W_2^{2l+1}, for any integer $l \geq 2$, are the odd holes and odd anti-holes, respectively.

In the remaining, let $\oplus$ stand for addition modulus n, i.e. $i \oplus j = (i+j) \mod n$ for $i, j \in \mathbb{Z}$.

Lemma 3 (Trotter (1975)) $\alpha(\overline{W}_p^n) = \omega(W_p^n) = \left\lfloor \frac{n}{p} \right\rfloor$ *and* $\alpha(W_p^n) = \omega(\overline{W}_p^n) = p$.



(a) W_3^8 (b) $\overline{W}_3^8$

Figure 1: Example of a web and an antiweb.

3.1 Web

We start by defining some stable sets of W_p^n. For each integer $i \geq 0$, define the following sequence of integers:

$$S_i = \langle i \oplus 0, i \oplus 1, \ldots, i \oplus (p-1) \rangle \tag{1}$$

Lemma 4 *For every integer $i \geq 0$, S_i indexes a maximum stable set of W_p^n.*

Proof: By the symmetry of W_p^n, it suffices to consider the sequence S_0. Let i and j be in S_0. Notice that $|i-j| \leq p-1 < p$. Then, $v_i v_j \notin E(W_p^n)$, which proves that S_0 indexes a stable set with cardinality $p = \alpha(W_p^n)$. □

Using the above lemma and sets S_i's, we can now calculate the k-th chromatic number of W_p^n. Our main ideia is to build a cover of the graph by stable sets, and show that each vertex of W_p^n is covered at least k times.

Theorem 1 *For every $k \in \mathbb{N}$, $\chi_k(W_p^n) = \left\lceil \frac{kn}{p} \right\rceil = \left\lceil \frac{kn}{\alpha(W_p^n)} \right\rceil$.*

Proof: By Lemma 2, we only have to show that $\chi_k(W_p^n) \leq \left\lceil \frac{kn}{p} \right\rceil$, for an arbitrary $k \in \mathbb{N}$. For this purpose, we show that $\Xi(k) = \langle S_0, S_p, \ldots, S_{(x-1)p} \rangle$ gives a k-fold x-coloring of W_p^n, with $x = \left\lceil \frac{kn}{p} \right\rceil$. We have that

$$\Xi(k) = \left\langle \underbrace{0 \oplus 0, 0 \oplus 1, \ldots, 0 \oplus p-1}_{S_0}, \underbrace{p \oplus 0, \ldots, p \oplus (p-1)}_{S_p}, \ldots, \underbrace{(x-1)p \oplus 0, \ldots, (x-1)p \oplus (p-1)}_{S_{(x-1)p}} \right\rangle.$$

Since the first element of $S_{(\ell+1)p}$, $0 \leq \ell < x-1$, is the last element of $S_{\ell p}$ plus 1 (modulus n), we have that $\Xi(k)$ is a sequence (modulus n) of integer numbers starting at 0. Also, it has $\left\lceil \frac{kn}{p} \right\rceil p \geq kn$ elements. Therefore, each element between 0 and $n-1$ appears at least k times in $\Xi(k)$. By Lemma 4, this means that $\Xi(k)$ gives a k-fold coloring of W_p^n with $\left\lceil \frac{kn}{p} \right\rceil$ colors, as desired. □

3.2 Antiweb

As before, we proceed by determining stables sets of the graph that cover each vertex of $\overline{W}_p^n$ at least k times. In order to index independent sets of $\overline{W}_p^n$, we define the sequences:

$$\begin{aligned} S_0 &= \left\langle \left\lceil t\frac{n}{\alpha(\overline{W}_p^n)} \right\rceil : t = 0,1,\ldots,\alpha(\overline{W}_p^n)-1 \right\rangle \\ S_i &= \langle j \oplus 1 : j \in S_{i-1} \rangle, \quad i = 1,2,3,\ldots \\ &= \langle j \oplus i : j \in S_0 \rangle, \quad i = 1,2,3,\ldots. \end{aligned}$$

The claimed property of each S_i will be shown with the help of the following lemmas.

Lemma 5 *If $x,y \in \mathbb{R}$ and $x \geq y$, then $\lfloor x-y \rfloor \leq \lceil x \rceil - \lceil y \rceil \leq \lceil x-y \rceil$.*

Proof: It is clear that $x - \lceil x \rceil \leq 0$ and $\lceil y \rceil - y < 1$. By summing up these inequalities, we get $\lfloor x-y+\lceil y \rceil - \lceil x \rceil \rfloor \leq 0$. Therefore, $\lfloor x-y \rfloor \leq \lceil x \rceil - \lceil y \rceil$. To get the second inequality, recall that $\lceil x-y \rceil + \lceil y \rceil \geq \lceil x-y+y \rceil = \lceil x \rceil$. □

Lemma 6 *For every antiweb $\overline{W}_p^n$ and every integer $k \geq 0$, $\left\lfloor \frac{nk}{\alpha(\overline{W}_p^n)} \right\rfloor \geq pk$.*

Proof: Since $\alpha(\overline{W}_p^n) = \left\lfloor \frac{n}{p} \right\rfloor$, we have that $\frac{n}{p} \geq \alpha(\overline{W}_p^n)$, which implies $\frac{nk}{\alpha(\overline{W}_p^n)} \geq pk$. Since pk is integer, the result follows. □

Lemma 7 *For $\overline{W}_p^n$ and every integer $l \geq 1$, $\left\lceil \frac{ln}{\alpha(\overline{W}_p^n)} \right\rceil - \left\lceil \frac{(l-1)n}{\alpha(\overline{W}_p^n)} \right\rceil \geq p$.*

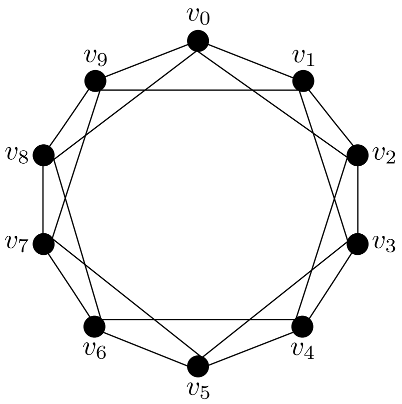
Proof: By Lemma 5, we get $\left\lceil \frac{ln}{\alpha(\overline{W}_p^n)} \right\rceil - \left\lceil \frac{(l-1)n}{\alpha(\overline{W}_p^n)} \right\rceil \geq \left\lfloor \frac{ln}{\alpha(\overline{W}_p^n)} - \frac{(l-1)n}{\alpha(\overline{W}_p^n)} \right\rfloor = \left\lfloor \frac{n}{\alpha(\overline{W}_p^n)} \right\rfloor$. The statement then follows from Lemma 6. □

Lemma 8 *For every integer $i \geq 0$, the vertices indexed by S_i form a maximum independent set of $\overline{W}_p^n$.*

Proof: By the simmetry of an antiweb and the definition of the S_i's, it suffices to show the claimed result for S_0. Let i and j belong to S_0. We have to show that $p \leq |i-j| \leq n-p$. For the upper bound, note that $|i-j| \leq \left\lceil \frac{(\alpha(\overline{W}_p^n)-1)n}{\alpha(\overline{W}_p^n)} \right\rceil = \left\lceil n - \frac{n}{\alpha(\overline{W}_p^n)} \right\rceil$. Lemma 6 implies that this last term is no more than $\lceil n-p \rceil$, that is, $n-p$. On the other hand, $|i-j| \geq \min_{l \geq 1} \left(\left\lceil \frac{ln}{\alpha(\overline{W}_p^n)} \right\rceil - \left\lceil \frac{(l-1)n}{\alpha(\overline{W}_p^n)} \right\rceil \right)$. By Lemma 7, it follows that $|i-j| \geq p$. Therefore, S_0 indexes an independent set of cardinality $\alpha(\overline{W}_p^n)$. □

The above lemma is the basis to give the expression of $\chi_k(\overline{W}_p^n)$.

Lemma 9 *Let be given an antiweb $\overline{W}_p^n$ and a positive integer $k \leq \alpha(\overline{W}_p^n)$. The index of each vertex of $\overline{W}_p^n$ belongs to at least k of the sequences $S_0, S_1, \ldots, S_{f(k)}$, where $f(k) = \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil - 1$.*



(a) $\overline{W}_3^{10}$.

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7
$A(\ell,0)$	0	1	2	3	4	5	6	$\cdots$
$A(\ell,1)$	4	5	6	7	8	9	0	$\cdots$
$A(\ell,2)$	7	8	9	0	1	2	3	$\cdots$

$\ell = 1$

$\ell = 2$

(b) $C(1)$ in blue, $C(2)$ in red.

Figure 2: Example of a 2-fold 7-coloring of $\overline{W}_3^{10}$. Recall that $\alpha(\overline{W}_3^{10}) = 3$.

Proof: Let $\ell \in \{1,2,\ldots,k\}$ and $t \in \{0,1,\ldots,\alpha(\overline{W}_p^n)-1\}$. Define $A(\ell,t)$ as the sequence comprising the $(t+1)$-th elements of $S_0, S_1, \ldots, S_{f(\ell)}$, that is,

$$A(\ell,t) = \left\langle \left\lceil t\frac{n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus i : \ i = 0,1,\ldots, \left\lceil \frac{\ell n}{\alpha(\overline{W}_p^n)} \right\rceil - 1 \right\rangle.$$

Since $\ell \le \alpha(\overline{W}_p^n)$, $A(\ell,t)$ has $\left\lceil \frac{\ell n}{\alpha(\overline{W}_p^n)} \right\rceil$ distinct elements. Figure 2 illustrates these sets for $\overline{W}_3^{10}$.

Let $B(\ell,t)$ be the subsequence of $A(\ell,t)$ formed by its first $\left\lceil \frac{(\ell+t)n}{\alpha(\overline{W}_p^n)} \right\rceil - \left\lceil \frac{tn}{\alpha(\overline{W}_p^n)} \right\rceil \le \left\lceil \frac{\ell n}{\alpha(\overline{W}_p^n)} \right\rceil$ elements (the inequality comes from Lemma 5). In Figure 2(b), $B(1,t)$ relates to the numbers in blue whereas $B(2,t)$ comprises the numbers in blue and red. Notice that $B(\ell,t)$ comprises consecutive integers (modulus n), starting at $\left\lceil \frac{tn}{\alpha(\overline{W}_p^n)} \right\rceil \oplus 0$ and ending at $\left\lceil \frac{(\ell+t)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus (-1)$. Let $C(1,t) = B(1,t)$ and $C(\ell+1,t) = B(\ell+1,t) \setminus B(\ell,t)$, for $\ell < k$. Similarly to $B(\ell,t)$, $C(\ell,t)$ comprises consecutive integers (modulus n), starting at $\left\lceil \frac{(\ell+t-1)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus 0$ and ending at $\left\lceil \frac{(\ell+t)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus (-1)$. Observe that the first element of $C(\ell,t+1)$ is the last element of $C(\ell,t)$ plus 1 (modulus n). Then, $C(\ell) = \langle C(\ell,0), C(\ell,1), \ldots, C(\ell,\alpha(\overline{W}_p^n)-1)\rangle$ is a sequence of consecutive integers (modulus n) starting at the first element of $C(\ell,0)$, that is $\left\lceil \frac{(\ell-1)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus 0$, and ending at the last element of $C(\ell,\alpha(\overline{W}_p^n)-1)$, that is $\left\lceil \frac{(\alpha(\overline{W}_p^n)+\ell-1)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus (-1) = \left\lceil \frac{(\ell-1)n}{\alpha(\overline{W}_p^n)} \right\rceil \oplus (-1)$. This means that $C(\ell) \equiv \langle 0,1,\ldots,n-1\rangle$. Therefore, for each $\ell = 1,2,\ldots,k$, $C(\ell)$ covers every vertex once. Consequently, every vertex is covered k times by $C(1), C(2), \ldots, C(k)$, and so is covered at least k times by $S_0, S_1, \ldots, S_{f(k)}$. $\square$

Now we are ready to prove our main result for antiwebs.

Theorem 2 *For every $k \in \mathbb{N}$, $\chi_k(\overline{W}_p^n) = \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil$.*

Proof: By Lemma 2, we only need to show the inequality $\chi_k(\overline{W}_p^n) \leq \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil$, for an arbitrary $k \in \mathbb{N}$. First, assume that $k \leq \alpha(\overline{W}_p^n)$. By lemmas 8 and 9, it is straightforward that the stable sets $S_0, S_1, \ldots, S_{x-1}$, where $x = \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil$, induce a k-fold x-coloring of $\overline{W}_p^n$. If $k > \alpha(\overline{W}_p^n)$ then we can write k as $k = l\alpha(\overline{W}_p^n) + i$, for some integers $l \geq 1$ and $0 \leq i < \alpha(\overline{W}_p^n)$. Sets $S_0, \ldots, S_{n-1}$ used l times together with sets $S_0, \ldots S_{y-1}$, where $y = \left\lceil \frac{in}{\alpha(\overline{W}_p^n)} \right\rceil$, induce a coloring of $\overline{W}_p^n$ with $ln + \left\lceil \frac{in}{\alpha(\overline{W}_p^n)} \right\rceil = \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil$ colors. In this coloring, each vertex is colored $l\alpha(\overline{W}_p^n) + i = k$ times. Therefore, it is a k-fold x-coloring of $\overline{W}_p^n$. □

3.3 Fractional chromatic number

By their definitions, the fractional chromatic number and the k-th chromatic number of a graph G are related as follows:

$$\bar{\chi}(G) = \min\left\{\frac{\chi_k(G)}{k} : k \in \mathbb{N}\right\}. \tag{2}$$

This observation and Lemma 2 lead to the following already known inequality $\bar{\chi}(G) \geq \frac{n}{\alpha(G)}$. Using theorems 1 and 2, we can show that this lower bound is tight for webs and antiwebs.

Proposition 1 *If G is the graph W_p^n or $\overline{W}_p^n$ then $\bar{\chi}(G) = \frac{n}{\alpha(G)}$.*

Proof: By theorems 1 and 2, $\frac{\chi_k(G)}{k} \geq \frac{n}{\alpha(G)}$ for every $k \in \mathbb{N}$ and this bound is attained with $k = \alpha(G)$. Then, by equation (2), the claimed result follows. □

3.4 Tight bounds

In the two previous subsections, we have shown that the k-th chromatic number of web and antiwebs achieve the lower bound given in Lemma 2. Here, we show that some of these graphs also yield the bounds presented in Lemma 1.

Proposition 2 *Let G be the graph W_p^n or $\overline{W}_p^n$, $r = n \mod \alpha(G)$ and $k \in \mathbb{N}$. Then, $\chi_k(G) = k\chi(G)$ if, and only if, $r = 0$ or $k < \frac{\alpha(G)}{\alpha(G)-r}$.*

Proof: By theorems 1 and 2, $\chi_k(G) = k\chi(G)$ if, and only if, $\left\lceil \frac{kn}{\alpha(G)} \right\rceil = k\left\lceil \frac{n}{\alpha(G)} \right\rceil$, which is also equivalent to $\left\lceil \frac{kr}{\alpha(G)} \right\rceil = k\left\lceil \frac{r}{\alpha(G)} \right\rceil$. If $r = 0$, this equality trivially holds. Otherwise, $\left\lceil \frac{r}{\alpha(G)} \right\rceil = 1$ and so the equality is equivalent to $\frac{rk}{\alpha(G)} > k - 1$ or still $k < \frac{\alpha(G)}{\alpha(G)-r}$. □

Proposition 3 *Let G be the graph W_p^n or $\overline{W}_p^n$ and $k \in \mathbb{N}$. Then, $\chi_k(G) = k\omega(G)$ if, and only if, p divides n.*

Proof: Let $s = n \mod p$. Note that $n = \lfloor n/p \rfloor p + s = \omega(G)\alpha(G) + s$. By theorems 1 and 2, we get

$$\chi_k(G) = \left\lceil \frac{kn}{\alpha(G)} \right\rceil = k\omega(G) + \left\lceil \frac{ks}{\alpha(G)} \right\rceil.$$

Therefore, the result follows. □

If p divides n, so does $\alpha(W_p^n)$ and $\alpha(\overline{W}_p^n)$. In this case, which holds for all perfect and some non-perfect webs and antiwebs, the above lemmas guarantee an equality of the bounds from Lemma 1.

Corollary 1 *Let G be the graph W_p^n or $\overline{W}_p^n$ and $k \in \mathbb{N}$. Then, $k\omega(G) = \chi_k(G) = k\chi(G)$ if, and only if, p divides n.*

On the other hand, the same bounds are allways strict for the minimal imperfect graphs.

Corollary 2 *If G is an odd hole or odd antihole then $k\omega(G) < \chi_k(G) < k\chi(G)$, for every integer $k > 1$.*

Proof: Let us first show that $\chi_k(G) < k\chi(G)$. By Proposition 2, we have to show that $r := n \mod \alpha(G) \neq 0$ and $s := \frac{\alpha(G)}{\alpha(G)-r} \leq 2$. First, let G be an odd antihole. Then, $G = W_2^{2\ell+1}$ for some $\ell \geq 2$, which implies that $r = 1$ and $s = \frac{2}{2-1} = 2$. Now, let G be an odd hole. Then, $G = W_p^{2p+1}$ for some $p \geq 2$. We have that $r = 1$ and $s = \frac{p}{p-1} \leq 2$.

To show the other inequality, it suffices to use Proposition 3 and observe that $n \mod p \neq 0$ for odd holes and odd anti-holes. □

4 χ_k-critical web and antiwebs

A graph is said to be χ-*critical* if $\chi(G - v) < \chi(G)$, for all $v \in V(G)$. Note that, for every vertex v of a critical graph, there is always an optimal coloring such that the color of v is not assigned to any other vertex. Not surprisingly, critical subgraphs play an important role in the context of vertex coloring. They are the core of reduction procedures [Herrmann and Hertz (2002)] as well as they provide facet-inducing inequalities of the vertex coloring polytope [Campêlo, Corrêa, and Frota (2004)]. Odd holes and odd anti-holes are examples of critical graphs.

In this vein, we define a χ_k-*critical* graph as a graph G such that $\chi_k(G - v) < \chi_k(G)$, for all $v \in V(G)$. If this relation holds for every $k \in \mathbb{N}$, G is said to be χ_*-*critical*. Now we investigate these properties for webs and antiwebs.

For the trivial case where $p = 1$, it is clear that W_1^n is χ_*-critical and $\overline{W}_1^n$ is not χ_k-critical, for all $k \in \mathbb{N}$. Then, it remains to analyse the case where $p > 1$.

Lemma 10 *If G is W_p^n or $\overline{W}_p^n$ and $p > 1$ then $\alpha(G - v) = \alpha(G)$ and $\omega(G - v) = \omega(G)$, for all $v \in V(G)$.*

Proof: Lemmas 4 and 8 imply that every vertex belongs to a maximum stable set of G. Since $p > 1$, $V(G)$ is not a stable set. Therefore, there is always a maximum stable set of G that does not contain a specific vertex v. It follows that $\alpha(G - v) = \alpha(G)$. Then, the other equality is a consequence of $\alpha(G) = \omega(\bar{G})$. □

Lemma 11 *If G is W_p^n or $\overline{W}_p^n$, $p > 1$ and p divides n then $\chi_k(G - v) = \chi_k(G)$, for all $v \in V(G)$.*

Proof: Using Lemma 1, Corollary 1 and Lemma 10, we get

$$k\omega(G-v) \le \chi_k(G-v) \le \chi_k(G) = k\omega(G) = k\omega(G-v).$$

Therefore, equality holds everywhere in the above expression. □

Corollary 3 *Let $k \in \mathbb{N}$. If $p > 1$ and p divides n then W_p^n and $\overline{W}_p^n$ are not χ_k-critical.*

For the case where $\frac{n}{p} \notin \mathbb{Z}$, we separately analyse W_p^n and $\overline{W}_p^n$.

Lemma 12 *Let $k \in \mathbb{N}$ and $v \in V(\overline{W}_p^n)$. If p does not divide n then $\chi_k(\overline{W}_p^n - v) = \left\lceil \frac{k(n-1)}{\alpha(\overline{W}_p^n)} \right\rceil$ and, consequently, $\left\lfloor \frac{k}{\alpha(\overline{W}_p^n)} \right\rfloor \le \chi_k(\overline{W}_p^n) - \chi_k(\overline{W}_p^n - v) \le \left\lceil \frac{k}{\alpha(\overline{W}_p^n)} \right\rceil$.*

Proof: By lemmas 2 and 10, we have that $\chi_k(\overline{W}_p^n - v) \ge \left\lceil \frac{k(n-1)}{\alpha(\overline{W}_p^n)} \right\rceil$. Now, we claim that $\overline{W}_p^n - v$ is a subgraph of $\overline{W}_p^{n-1}$. First, notice that this antiweb is well-defined. Indeed, $n-1 \ge 2p$ because p does not divide n. Now, let $v_iv_j \in E(\overline{W}_p^n - v) \subset E(\overline{W}_p^n)$. Then $|i-j| > p$ or $|i-j| > n-p > (n-1)-p$. Therefore, $v_iv_j \in E(\overline{W}_p^{n-1})$. This proves the claim. Then, Theorem 1 implies that $\chi_k(\overline{W}_p^n - v) \le \chi_k(\overline{W}_p^{n-1}) = \left\lceil \frac{k(n-1)}{\alpha(\overline{W}_p^{n-1})} \right\rceil$. Moreover, since p does not divide n, it follows that $\alpha(\overline{W}_p^{n-1}) = \left\lfloor \frac{n-1}{p} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor = \alpha(\overline{W}_p^n)$. This shows the converse inequality $\chi_k(\overline{W}_p^n - v) \le \left\lceil \frac{k(n-1)}{\alpha(\overline{W}_p^n)} \right\rceil$.

To get the second part of the statement, it suffices to use Lemma 5 and the expressions of $\chi_k(\overline{W}_p^n)$ and $\chi_k(\overline{W}_p^n - v)$. □

Corollary 4 *Suppose that p does not divide n. Then, $\overline{W}_p^n$ is χ_k-critical if, and only if, $\left\lceil \frac{k(n-1)}{\alpha(\overline{W}_p^n)} \right\rceil < \left\lceil \frac{kn}{\alpha(\overline{W}_p^n)} \right\rceil$. In particular, $\overline{W}_p^n$ is χ_k-critical for all integer $k \ge \left\lfloor \frac{n}{p} \right\rfloor$.*

To conclude this section, we consider the case where $p > 1$ and $\frac{n-1}{p} \in \mathbb{Z}$, which includes holes and antiholes.

Lemma 13 *Let $k \in \mathbb{N}$ and $v \in V(W_p^n) = V(\overline{W}_p^n)$. If $p > 1$ and p divides $n-1$ then $\chi_k(\overline{W}_p^n - v) = kp$ and $\chi_k(W_p^n - v) = \frac{k(n-1)}{p}$.*

Proof: Assume that $p > 1$ and $\frac{n-1}{p} \in \mathbb{Z}$. Then, $\frac{n}{p} \notin \mathbb{Z}$ and $\alpha(\overline{W}_p^n) = \frac{n-1}{p}$. By Lemma 12, $\chi_k(\overline{W}_p^n - v) = kp$. For a web, we can use lemmas 2 and 10 to get that $\chi_k(W_p^n - v) \ge \left\lceil \frac{k(n-1)}{\alpha(W_p^n)} \right\rceil = \frac{k(n-1)}{p}$. By the symmetry of W_p^n, we only need to prove the converse inequality for $v = v_{n-1}$. Let us use (1) to define $\Xi' = \langle S_0, S_p, \ldots, S_{(\frac{n-1}{p}-1)p} \rangle$. We can see that Ξ' is the sequence $\langle 0, 1, \ldots, n-1 \rangle$. Therefore, Ξ' gives an 1-fold $\left(\frac{n-1}{p}\right)$-coloring of $W_p^n - v_{n-1}$. It follows that $\chi_k(W_p^n - v) \le \frac{k(n-1)}{p}$. □

Corollary 5 *If $p > 1$ and p divides $n-1$ then W_p^n and $\overline{W}_p^n$ are χ_*-critical. In particular, the odd holes and odd antiholes are χ_*-critical.*

Proof: Let $k \in N$. For a web, it is clear that $\chi_k(W_p^n) = \left\lceil \frac{kn}{p} \right\rceil \geq \frac{kn}{p} > \frac{k(n-1)}{p} = \chi_k(W_p^n - v)$. For an antiweb, since $\alpha(\overline{W}_p^n) = \frac{n-1}{p}$ under the hypothesis, it follows that $\chi_k(\overline{W}_p^n) = \left\lceil \frac{kpn}{n-1} \right\rceil \geq \frac{kpn}{n-1} > kp = \chi_k(\overline{W}_p^n - v)$. In both cases, we get the condition for being χ_*-critical. $\square$

5 Conclusion and Future Work

Vertex coloring is a covering of vertices by stable sets. One measure associated with vertex coloring is the k-th chromatic number. For $k = 1$, this is exactly the classical chromatic number.

An important class of graphs in the context of stable sets and, consequently, in the context of coloring is the webs and antiwebs. For instance, it is known that these structures induce facets of the stable set and the vertex coloring polytopes [Trotter (1975); Cheng and Vries (2002a, 2002b); Palubeckis (2010)]. The expression of these facets depends on the stability and the chromatic numbers, which are known for webs and antiwebs.

A natural question then arises concerning the role of webs and antiwebs play in the problem of finding the k-th chromatic number of a graph. In this context, determining the exact value of k-th chromatic number of webs and antiwebs is a primary ingredient. We successfully tackled this problem. Some basic consequences of this result are also investigated. Particularly, we identified all the antiwebs and some webs that are χ_k-critical.

We intend to expand this work by characterizing all χ_k-critical webs as well as investigating the importance of χ_k-critical webs and antiwebs to charactering the facial structure of the polytope associated with the k-th chromatic number problem.

References

Campêlo, M., Corrêa, R., & Frota, Y. (2004). Cliques, holes and the vertex coloring polytope. *Information Processing Letters*, *89*(4), 159–164.

Cheng, E., & Vries, S. de. (2002a). Antiweb-wheel inequalities and their separation problems over the stable set polytopes. *Mathematical Programming*, *92*, 153-175.

Cheng, E., & Vries, S. de. (2002b). On the facet-inducing antiweb-wheel inequalities for stable set polytopes. *SIAM Journal on Discrete Mathematics*, *15*(4), 470-487.

Chudnovsky, M., Robertson, N., Seymour, P., & Thomas, R. (2006). The strong perfect graph theorem. *Annals of Mathematics*, *164*, 51–229.

Geller, D., & Stahl, S. (1975). The chromatic number and other functions of the lexicographic product. *Journal of Combinatorial Theory*, *19*, 87–95.

Halldórsson, M., & Kortsarz, G. (2002, February). Tools for multicoloring with applications to planar graphs and partial k-trees. *Journal of Algorithms*, *42*, 334–366.

Herrmann, F., & Hertz, A. (2002). Finding the chromatic number by means of critical graphs. *ACM Journal of Experimental Algorithmics*, *7*.

Holm, E., Torres, L. M., & Wagler, A. K. (2010). On the chvátal-rank of antiwebs. *Electronic Notes in Discrete Mathematics*, *36*, 183 - 190. (ISCO 2010 - International Symposium on Combinatorial Optimization)

Narayanan, L. (2002). *Channel assignment and graph multi-coloring*. Wiley.

Opsut, R. J., & Roberts, F. S. (1981). On the fleet maintenance, mobile radio frequency, task assigment and traffic phasing problems. *The Theory and Applications of Graphs*, 479–492.

Palubeckis, G. (2010). Facet-inducing web and antiweb inequalities for the graph coloring polytope. *Discrete Applied Mathematics*, *158*, 2075–2080.

Pêcher, A., & Wagler, A. K. (2006). Almost all webs are not rank-perfect. *Mathematical Programming*, *105*, 311-328.

Ren, G., & Bu, Y. (2010, October). *k*-fold coloring of planar graphs. *Science China Mathematics*, *53*(10), 2791–2800.

Roberts, F. S. (1979). On the mobile radio frequency assignment problem and the traffic light phasing problem. *Annals of NY Academy of Sciences*, *319*, 466–483.

Scheinerman, E. R., & Ullman, D. H. (1997). *Fractional graph theory: A rational approach to the theory of graphs* (1st ed., Vol. 1). Wiley-Interscience.

Stahl, S. (1976). *n*-tuple colorings and associated graphs. *Journal of Combinatorial Theory*, *20*, 185–203.

Trotter, L. E. (1975). A class of facet producing graphs for vertex packing polyhedra. *Discrete Mathematics*, *12*, 373–388.

Wagler, A. K. (2004). Antiwebs are rank-perfect. *4OR: A Quarterly Journal of Operations Research*, *2*, 149-152.