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# Constraint Programming / Mathematical Programming Hybrid Methods for Combinatorial Optimization

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# History

- Issued from Artificial Intelligence
- First Langage of Constraint Programming in 1963 (I. Sutherland)
- First application in O.R. in the 70s
- Great Success in Planning, Scheduling, ...

# How does it works ?

- CP was first designed for solving Satisfiability Problems: Find a value for each variable $x_i$ $(i = 1, \ldots, n)$ such that

$$\begin{cases} C_j(x) \text{ is true } \forall j = 1, \ldots, m \\ x_i \in D_i \ \forall i = 1, \ldots, n \end{cases}$$

- $D_i$ is the set of possible values for $x_i$
- $C_j$ are constraints on variables $x_i$

# How does it works ?

- $D_i$ is called the domain of $x_i$
- To each of the constraints $C_j(x)$ is associated a Filtering Algorithm which aims to remove values of $D_i$ which are not possible anymore with respect to the constraint.
- Example: $3x_1 + 5x_2 + 7x_3 \leq 6 \qquad x \in \{0,1\}^3$

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$$\Longrightarrow x_3 = 0$$

# Propagation

- Once a filtering algorithm succeeded in removing a value for a given variable, the filtering algorithms of the constraints containing this variable are successively called.
- This is the propagation phase

# Example

- $$\begin{cases} C_1 : & 6x_1 + 4x_2 + 4x_3 \geq 5 \\ C_2 : & 3x_1 + 5x_2 + 7x_3 \leq 6 \\ & x \in \{0,1\}^3 \end{cases}$$

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A solution has been found !

# How does it works ?

- If the propagation phase does not succeed in neither finding a solution nor proving that there is no solution, then a search tree is developed.
- It is based on a partition of the feasible set.

# CP for Optimization

- $\min\{f(x)/C_j(x) \text{ is true } j = 1, \ldots, m \; x_i \in D_i \; i = 1, \ldots, n\}$

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- A filtering algorithm has to be designed for the objective function.
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  - The domain of $Z$ by reducing the bounds.
  - $D_i$ by removing values which are not consistent with the current upper and lower bounds.

# Filtering Algorithm

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  - $x_i \in [l_i, u_i]$ and integer.

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- If $a_1 \geq 0$ and $M < u_1$ then $u_1 = M$

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- It happens that more than one filtering algorithm are associated to the same constraint and are succesively called if they make different deductions ...
- It also happens that different version of the same algorithm, more or less powerful but more or less expensive to run, are considered and called appropriately according to some criteria.

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- Choose a constraint $C$ in $\mathcal{L}$
- Call the Filtering Algorithm of $C$
- For each variable $v$ whose domain has been reduced:
  - Add to $\mathcal{L}$, all the constraints where $v$ is involved and which are not already in $\mathcal{L}$.

# Available tools

- CP-Optimizer from IBM (ex-ILOG), works like CPLEX (CONCERT etc.). Very expensive but free for academics !

  www-01.ibm.com/softwre/commerce/optimization/cplex-cp-optimizer/

- CHOCO, open-source.

  www.emn.fr/z-info/choco-solver/

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- Constraints might include disjunctions, implications, etc.
- Variables can be of any type (real, integer, set, object of any type)

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- Filtering algorithms taking into account a whole set of constraints have been developed.
- The whole set of constraints is then contracted in a unique constraint, called Global Constraint.

# The AllDifferent Global Constraints

- AllDifferent should be used when n variables have to take a different value.
- $AllDifferent(x) \iff x_i \neq xj \qquad \forall 1 \leq i < j \leq n$
- J-Ch Regin (94) designed a very efficient filtering algorithm for this global constraint.

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$AllDifferent(x) \implies x_1 = 3\ x_3 = 4.$

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- and ... an exponential number of constraints to avoid subcycles.

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- Only one constraint !
- That does not make the problem easier !

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This is a cooperative scheme !!!

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- Sometimes, the goods can be only delivered in a time window: this is the VPRTW.
- According to the context, additional constraints can be considered such as vehicle capacities.

# Modelling with CP : the VRPTW

- data : $n$ customers, $C_{ij}$: travel cost from $i$ to $j$, $T_{ij}$: travel time from $i$ to $j$, $[a_i, b_i]$: time window of $i$, $r_i$: demand of $i$, $K$: capacity of the vehicles, $v$: number of vehicles.

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- All the vehicles are used and a customer is served by exactly one vehicle.
- A vehicle can arrive before the time window, but it will have to wait.

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\begin{array}{ll}
\min & \sum_{i=1}^{n} C_{i,S_i} \\
\text{s.t.} & AllDifferent(S) \\
& S_i = j \Rightarrow l_j = l_i - r_i \ \forall i,j \\
& S_i = j \Rightarrow t_j \geq t_i + T_{ij} \ \forall i,j \\
& S_i \in \{1,\ldots,n\}/\{i\} \ \forall i \\
& l_i \in [0,K] \ \forall i \\
& t_i \in [a_i,b_i] \ \forall i
\end{array}
$$

# Modelling with CP : the VRPTW

Such a model has been used for the desing of heuristics for the VRPTW by :

- Caseau-Laburthe 99
- Feillet et al 01
- Pesant et al 98
- Rousseau 02

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# The Cumulative Global Constraint

- Scheduling problems consists in finding when (and, sometimes, where) a set of activities has to be executed.
- Each activity $i$ requires a quantity $r_{i,k}$ of a given resource $k$ which cannot provide more than $R_k$ units.
- Hence, at each time of the planning period, the set of activities currently performed must respect the capacity of each of the resources.

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  - Edge Finding (Carlier-Pinson 90)

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  - $S = (S_1, S_2, \ldots, S_n)$ the vector of starting times
- $Cumulative(S, d, r, R) \iff \sum_{\{i \;\; S_i \leq t < S_i + d_i\}} r_i \leq R \; \forall \, t$

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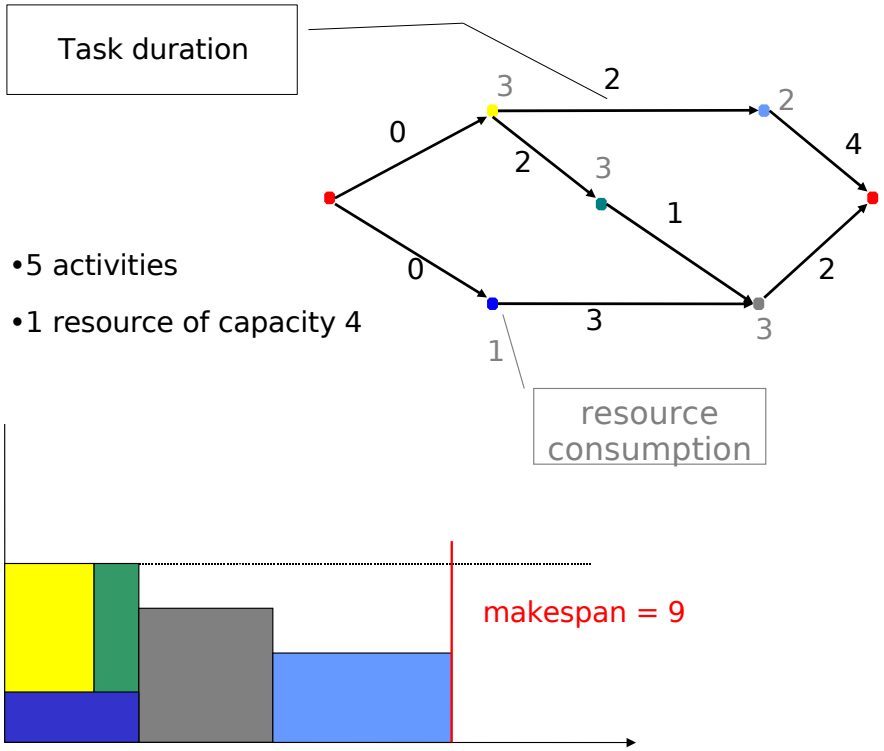
- The resources are renewable.
- The activities are not preemptive.
- The objective is to minimize the makespan of the project.

# Modelling with CP: the RCPSP




- 5 activities
- 1 resource of capacity 4

# Modelling with CP: the RCPSP

- $K$ resources of cap. $R_k$.
- $n$ activities of duration $d_i$ and resource consumption $r_{ik}$.
- Precedence constraints G=(V,E).
- Decision variables: $S_i$, Starting time of $i$, $C_{max}(= S_{n+1})$, total duration of the project.

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$$
\begin{array}{lll}
\min & C_{max} & \\
\text{s.t.:} & C_{max} \leq S_i + d_i & \forall i \\
 & S_j \geq S_i + d_i & \forall (i,j) \in E \\
 & Cumulative(S, d, r_{.k}, R_k) & \forall k \\
 & S \geq 0 &
\end{array}
$$

# Solving the RCPSP with a pure CP approach

- The filtering algorithm associated with the precedence constraints is a longest path algorithm.
- Many authors solved the RCPSP with a pure CP approach:
  - Klein-Scholl 99
  - Baptiste-Le Pape 00
  - Liess-Michelon 03
  - Etc.

# The Railway Rescheduling Problem

- Joint work with R. Acuña-Agost, S. Gueye, and D. Feillet.
- CP-AI-OR proceedings, Lecture Notes in Computer Sciences, V. 5547, 312-313, 2009.
- Deals with the the impact of disruptions in order to ensure the best possible service for the passengers.

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- It considers both Timetabling and Platforms Assignment Problems.
- Input : Original Plan and the incidents (with the associated delays).

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These objectives are reduced to only one objective by a weighted linear combination.

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- Since Constraints Programming Models are "smaller", it may be usefull to try to use CP ...

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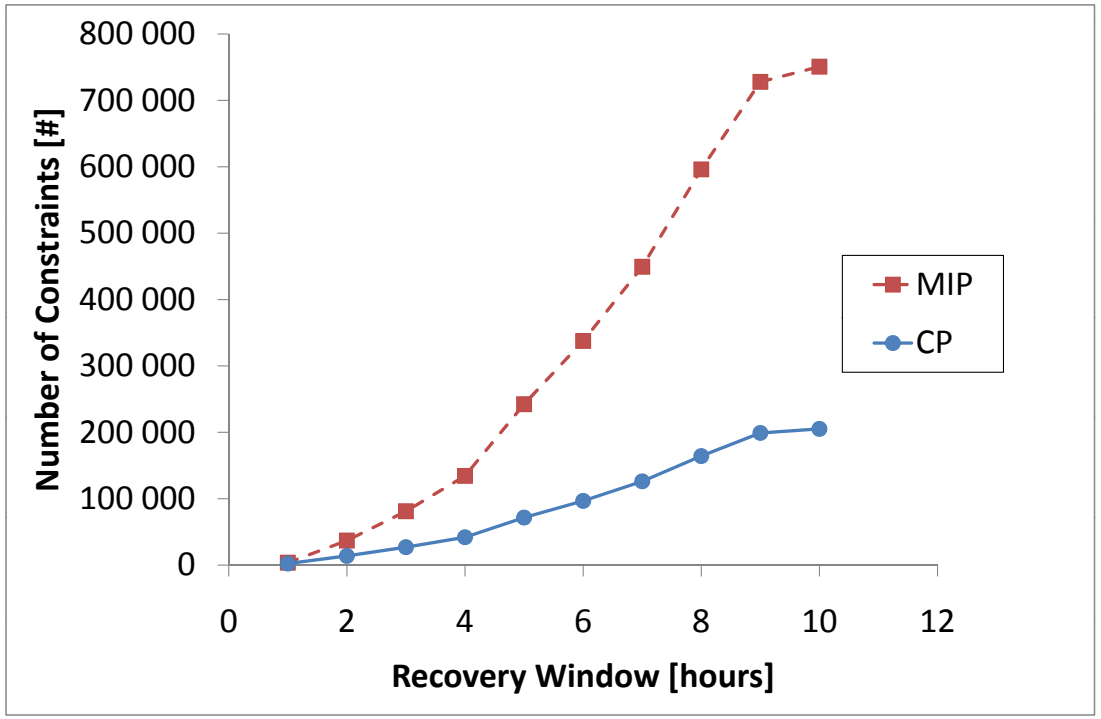
- $\lambda_{k\hat{k}} = 1$ if event $k$ occurs after event $\hat{k}$, not as planned originally.

# Constraint Programming Decisions Variables

- $Delay_{in}$ : Delay of train $i$ at node $n$.
- $Start_{in}$ : Arrival time of train $i$ at node $n$.
- $Track_{in}$ : Track used by train $i$ at node $n$.
- $UP_{in}$ : 1, if train $i$ performs an unplanned stop in the node $n$; 0 otherwise.

# Comparison of the Models



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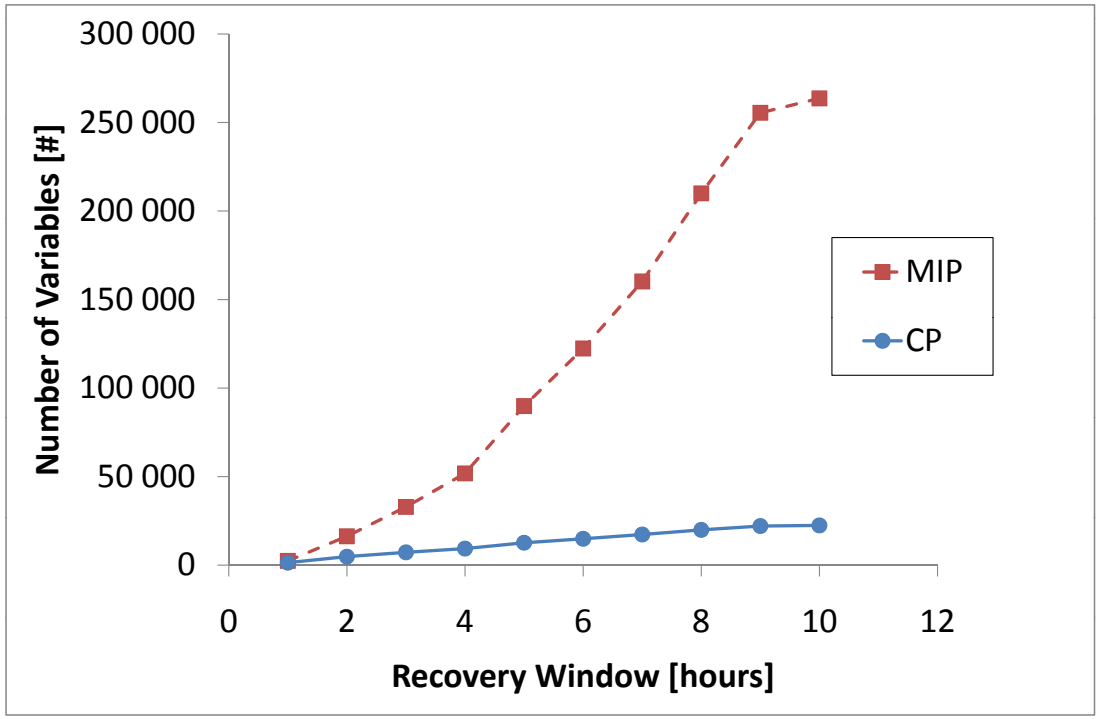

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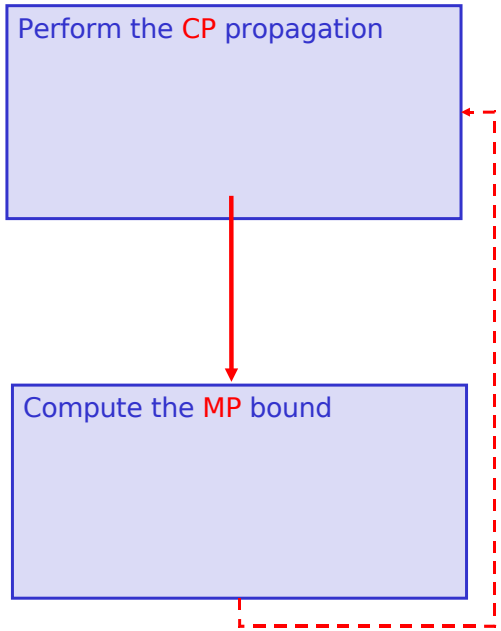
- They are complementary methods !
- Cooperative schemes should be useful !

# Cooperative Schemes

- The first cooperative schemes to appear in the literature were based on Mathematical Programming decomposition.
- Benoit et al 02 solved with a pure CP approach the master problem in a Bender's decomposition approach.
- Rousseau et al 02 solved with a pure CP approach the sub problem in a column generation method.

# Cooperative Schemes



# The Reduced Cost Constraint

- $$\begin{array}{ll} \min & Z = \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} & \sum_{i=1}^{n} a_{j,i} x_i + s_j = b_j \\ & x \in \{0,1\}^n \\ & s \geq 0 \end{array}$$

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- At the optimum of the linear relaxation, the objectif function is expressed as:

$$Z = lb + \sum_{i \in N_0} \overline{c}_i x_i + \sum_{i \in N_1} \overline{c}_i (1 - x_i) + \sum_j \overline{d}_j s_j$$

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  - find some upper bounds on the slack variables.

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An upper bound $u_j$ over the slack variable allows to transform the initial constraint into:

$$u_j \leq \sum_{i=1}^{n} a_{j,i} x_i \leq b_j$$

# The Multi-Knapsack Problem

- Generalisation of the classical knapsack Problem.

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.:} & Ax \leq b, \\ & x \in \{0,1\}^n, \end{array}$$

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- where $A \geq 0^{m \times n}, b \geq 0^m, c \geq 0^n$

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- Joint work with C.D. Rodrigues and M. Campelo, published in the CP proceeding, 08.

# Application to the Multi-Knapsack Problem

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3500

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- Let $\overline{k}$ and $\underline{k}$ be the optimal values of the above problem.

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- For $k \in \{\underline{k}, \ldots, \overline{k}\}$, create a subproblem :

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- Apply an Hybrid CP/MP method to each of them.

# The KnapSum Constraint

- At each node of the search tree, the CP lawyer has to takcle constaints of the forme :$\sum_{i=1}^{n} x_i = k$ and $\sum_{i=1}^{n} a_{ji} x_i \leq \bar{b}_j$

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- $$knapSum(a_j, b_j, k, x) \Leftrightarrow \underline{b_i} \leq \sum_{i=1}^{n} a_{ji} x_i \leq \bar{b}_i; \quad \sum_{j=1}^{n} x_j = k$$

# The KnapSum Constraint Filtering Algorithm

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  - Set $x_1$ to 0, and check if $\bar{S} < \underline{b_i} \Rightarrow$ Fix $x_1 = 1$

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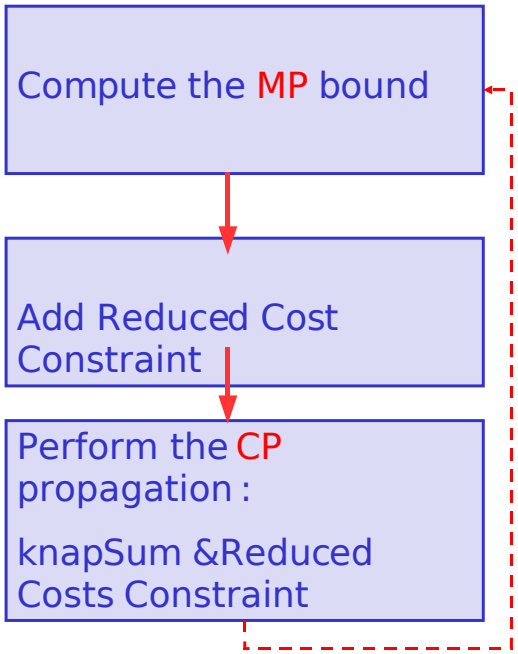
# Summary of the Hybrid Method for the MKP



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- ... but could be useful for Constraint Programming.

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- These aggregated constrainst will only be used in the CP layer of our process.
- Several aggregations were tested but the most efficient was to aggregate each of intial constraints with a constraint requiring that the objective function to be greater than the best known solution.

# Numerical Results for the MKP

- We compared our methods with two other methods: CPLEX and the algorithm presented in Vimont, Boussier and Vasquez [2007].

Table : Average time for each instance set in seconds

| Instance Set | CPLEX | VBV | KPS | KPS + Combo |
|---|---|---|---|---|
| 5.100 | 5.16 | 0.19 | 0.06 | 0.05 |
| 5.250 | 244.02 | 99.73 | 1.5 | 1.23 |
| 5.500 | 4219.56 | 926.7 | 19.09 | 14.35 |
| 10.100 | 39.53 | 6.37 | 2.54 | 2.07 |
| 10.250 | N/A | 8536.26 | 3050.51 | 2392.99 |
| 30.100 | 3469.27 | 1376.09 | 587.5 | 380.83 |

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- ... but they can still be improved.
- People working with "pure" CP approach have developped very efficient rules and techniques for "branching".
- It might be an interesting and fruitfull research direction to see how update these rules for "pure" IP or Hybrid approach.

THAT'S IT !

# Something else ...

- The Universities of Avignon, Federal of Ceará (Manoel Camplo), Federal of Rio de Janeiro (Nelson Maculan) and Federal Fluminense (Luiz Satoru Ochi) are collaborating in a PVE ("Ciência sem Fronteira") project.

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- We are looking for good candidates ...
- Besides this program, I am also looking for Ph.D. students who would like to make an entire Ph.D. in Avignon.