# A strong mixed integer formulation for a switch allocation problem

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**Abstract.** The switch allocation problem encompasses a set of combinatorial optimization problems faced by power utilities. The problem considered in this paper has the objective to determine the optimal locations for automatic switches in a primary distribution network. The role of an automatic switch is to enable the network reconfiguration so that faults can be isolated once they occur. Finding the optimal locations to install automatic switches has the potential to reduce operational costs and increase the expected reliability for the customers. This paper presents a flow-based mixed integer linear programming model and a set of strong valid inequalities. Computational experiments solve a benchmark of distribution networks using a branch-and-cut model embedded with the proposed valid inequalities. The results confirm the strength of the model by providing optimal solutions for all instances.
**KEYWORDS: branch-and-cut, power distribution networks, reliability optimization**

## 1 Introduction

Approximately 70% of the overall interruption duration of power systems is associated with contingencies in primary distribution networks (Billinton and Allan 1996). A distribution system consists of a set of components that are subject to failure, including cables, poles, breakers, switches, transformers, capacitors, and voltage regulators. During a contingency, parts of the network are disconnected from the substation until a maintenance team is able to identify the source of the failure and repair the damaged component. During this process, some customers are left without power, which negatively impacts the system reliability. A proper allocation of switches can reduce the impact of the outages by isolating the upstream customers from the fault.

The switch allocation problem (SAP) can be stated as the problem of finding the best locations of switches to be installed on a distribution network. The potential benefits of switch allocation include reducing the average duration of failures, improving the quality of the power supply and avoiding fines related to the violation of reliability standards. This work considers the allocation of automatic sectionalizers (normally closed switches) to minimize the expected energy not supplied.

Many heuristics have been proposed to solve the switch allocation problem. Levitin et al. (1995) proposed a genetic algorithm to allocate sectionalizing switches on a network with 52 nodes. Billinton and Jonnavithula (1996) used simulated annealing to determine the number and locations of switches for a network with 7 feeders and 67 nodes. Moradi and Fotuhi-Firuzabad (2008) proposed a particle swarm optimization algorithm to allocate sectionalizers and breakers for an IEEE feeder with 123 nodes.

Carvalho et al. (2005) addressed the allocation of automatic switches. A three step heuristic is proposed: *(i)* evaluate the benefit brought by each switch; *(ii)* partition the network into independent subsets; *(iii)* allocate the switches on each partition. This heuristic was tested on a network with 11 candidate locations to install switches. Benavides et al. (2013) also allocate automatic switches and propose an iterated sample construction with path relinking to solve networks up to 873 nodes. Optimal solutions were obtained for instances with 10 switches and 135 nodes. A memetic algorithm, proposed by Assis et al. (2014), considers sectionalizing and tie switches of different capacities, with manual or automatic operation schemes. Their approach minimizes the costs of allocation and energy not supplied, under reliability and flow capacity constraints. Comparing the efficiency of different methods is not straightforward, as Benavides et al. (2013) well observed, and the reason is the use of different objectives and reliability indices.

An exact algorithm, based on dynamic programming, was presented by Celli and Pilo (1999). However, the methodology is only applicable to small problems, such as the illustrative cases with 36 and 47 nodes, presented in their paper.

The main contribution of this paper is to propose an MILP (mixed integer linear programming) model with strong valid inequalities for the switch allocation problem. This combinatorial optimization problem has the objective of finding the minimum energy not supplied on a power distribution network through the allocation of automatic switches. The correctness of the MILP model is supported by an analytical reliability evaluation of the network (Usberti et al. 2013), performed through a set of variables called *interruption time flows*.

This paper is organized as follows. The introductory aspects are described in Sec. 2. The procedure to determine the interruption time of each customer in the network is described in Sec. 3. The concept of interruption time flows is introduced in Sec. 4, setting the background for an analytical reliability evaluation methodology. Sec. 5 gives numerical examples to illustrate the approach, and the MILP model for the switch allocation problem is presented in Sec. 6. Valid inequalities based on the interruption time flows are proposed and discussed on Sec. 7. Computational tests are performed on a benchmark of distribution networks in Sec. 8, where the proposed model is solved by a state-of-the-art optimization solver. Final remarks and conclusions enclose this paper in Sec. 9.

## 2 Reliability evaluation of power distribution systems

This section discusses an analytical reliability evaluation framework for radially operated power distribution systems. The methodology, proposed by Usberti et al. (2013), was adapted for the consideration of automatic switches in the network.

### 2.1 Assumptions

The proposed evaluation procedure adopts the following assumptions:

- The distribution network is radially operated, meaning there is an unique path linking the substation to each customer.
- All failures are non-transient short circuits that will propagate upstream until it finds an automatic switch.
- All failures are independent from each other and at most one failure happens at any time.
- The failure frequency is a stochastic variable, and its value represents the expected amount of failures that should occur in a one year period. This extends to all variables and indices that depend on the failure frequency.
- Automatic switches do not fail.

### 2.2 Main concepts

A radially operated distribution system, using graph terminology, can me modeled as a connected directed tree $T(V,A)$, rooted at the substation (node 0). Node $i \in V$ denotes either a network bifurcation point or a load point with power load $l_i$ $(kW)$, failure rate $\lambda_i$ (failures/year) and number of customers $n_i$. An arc $(i,j) \in A$, $i,j \in V$, is orientated in the same direction as the power flow, which traverses from the root to the customers. Each node $j \in V \setminus \{0\}$ has a predecessor node $i$, or simply, $i = pred(j)$. The set of arcs in which a switch is installed is denoted by $A_s$.

There is an unique directed path connecting the root to every node in the tree. The set of nodes (and arcs) representing the directed path connecting two nodes $i$ and $j$ is represented by $path(i,j)$. If no such path exists, then $path(i,j) = \emptyset$; also, $path(i,i) = \{i\}$. For every pair of nodes $i$ and $j$, if $path(i,j) \neq \emptyset$, then $j$ is downstream of $i$, otherwise $j$ is upstream of $i$. If an arc $(i',j') \in path(i,j)$ then $i'$ and $j'$ are consecutive nodes in that path. The set of downstream nodes of $i$ is represented by

$V_i$. If $V_i = \{i\}$ then node $i$ is a leaf. Eq. (1) shows how to determine the downstream power load $\tilde{l}_i$ of a node $i$.

$$\tilde{l}_i = \sum_{j \in V_i} l_j \tag{1}$$

### 2.3 Role of automatic switches

An automatic sectionalizer is a normally closed switch with an important role on the reliability of distribution networks. This switch opens immediately after a short circuit flows through it, in the intent to isolate the failure. This event disconnects all downstream loadpoints but it also prevents power interruption of any upstream load point. The time required to restore power supply to all customers affected by a fault in node $i$ is given by parameter $t_i$, called average interruption time. This parameter considers identifying the failure location, setting the maintenance team and repairing all defective network components. Only then the automatic switch can be reclosed. IEEE Std 493-1997 (1998) shows how to estimate the average interruption time and interruption frequency of a distribution network.

In the illustrative network shown in Figure 1 suppose that a fault occurs at node 2. This opens the switch on arc $(1,2)$, interrupting node 2 power supply. Suppose now that the fault occurs at node 3. In this case, the short circuit propagates up to the substation (which usually contains a circuit breaker), causing the interruption of all nodes.

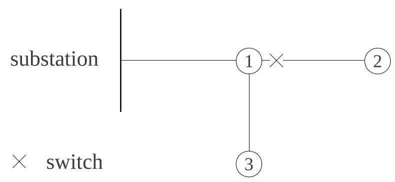



**Fig. 1.** Role of an automatic switch.

### 2.4 Reliability indices

To evaluate quantitatively how reliable a distribution system is, there are several indices that can be used. Some examples are the *system average interruption frequency index* (SAIFI), *system average interruption duration index* (SAIDI) and *energy not supplied* (ENS) (Eq. 2). These indices are not deterministic values but expectations of a probability distribution (Billinton and Allan 1996).

$$SAIFI = \frac{\sum_{i \in V} n_i \lambda_i}{\sum_{i \in V} n_i}, \qquad SAIDI = \frac{\sum_{i \in V} n_i u_i}{\sum_{i \in V} n_i}, \qquad ENS = \sum_{i \in V} l_i u_i \tag{2}$$

Parameters $n_i$, $\lambda_i$ and $l_i$ are, respectively, the number of customers, failure rate $(failures/year)$ and power load $(kW)$ of node $i$. The interruption times $u_i$ are variables discussed in Sec. 3.

Regulatory agencies adopt some of these indices to define minimum levels of reliability that, if not regarded, can trigger costly fines to the utilities. Moreover, reliability indices can also be used to $(i)$ identify areas of the network that require more investment; $(ii)$ determine the reliability tendency over time; $(iii)$ compare historical values with the current network state; $(iv)$ compute the benefit/loss of any proposed change to the network (Brown 2008).

## 3 Interruption times

The expected time each node will be interrupted due to faults occurred on its own components is the subject of Def. 1.

**Definition 1.** *The self-interruption time $\theta_i$ (Eq. 3) is the expected duration of interruptions from power supply a node $i$ will endure in a one-year period, due to local faults (occurred at node $i$).*

$$\theta_i = \lambda_i t \tag{3}$$

Def. 2 describes how to compute downstream interruption times.

**Definition 2.** *The downstream interruption time $\tilde{\theta}_i$, determined by Eq. (4), is the expected time node $i$ will be interrupted due to downstream faults, in a one-year period.*

$$\tilde{\theta}_i = \theta_i + \sum_{(i,j)\in A\setminus A_s} \tilde{\theta}_j \tag{4}$$

The downstream interruption time follows the functionality of switches described in Sec. 2.3. Switches avoid the accumulation of downstream interruption times.

The interruption time $u_i$ of a node $i$ is described by Def. 3.

**Definition 3.** *The interruption time $u_i$ is the expected duration of interruptions from power supply a node $i$ will endure in a one-year period, due to all faults occurred in the network. This variable can be calculated recursively through Eq. (5).*

$$u_0 = \tilde{\theta}_0, \qquad u_j - u_i = \begin{cases} 0 & (i,j) \in A \setminus A_s \\ \tilde{\theta}_j & (i,j) \in A_s \end{cases} \tag{5}$$

## 4 Flow-based reliability evaluation

The new reliability evaluation procedure proposed in this paper uses the concept of interruption time flows $f$ (Def. 4). This section shows that reliability indices, such as the ENS, can be expressed in terms of the $f$ variables (Theo. 1). The proposed methodology represents a less natural approach than using the interruption times $u$. However, the interruption time flows turn the reliability evaluation into a network flow problem that is more suitable to formulate within related optimization problems (Sec. 6).

**Definition 4.** *The interruption time flow $f_{ij}$ (Eq. 6), from node $j$ to node $i$ (inverse of the arc orientation), is the expected time node $i$ will be interrupted due to faults originated at nodes $k \in V_j$.*

$$f_{ij} = \begin{cases} 0 & (i,j) \in A_s \\ \tilde{\theta}_j & (i,j) \in A \setminus A_s \end{cases} \tag{6}$$

The presence of switches prevents the usual input equals output flow conservation laws for each node. To capture the flow deviation of the nodes, an interruption residue $F$ is defined next.

**Definition 5.** *The interruption residue $F_j$ represents an interruption flow balance of node $j$, calculated according to Eq. (7).*

$$F_j = \begin{cases} \theta_0 + \sum_{(0,k)\in A} f_{0k} & j = 0 \\ \theta_j + \sum_{(j,k)\in A} f_{jk} - f_{ij} & (i,j) \in A \end{cases} \tag{7}$$

Lem. 1 and Lem. 2 show that the interruption times $u$ can be determined by an expression of the interruption residues $F$.

**Lemma 1.** *The interruption time $u_j$ of a node $j$ can be determined recursively according to Eqs.* (8) *and* (9)*.*

$$u_0 = F_0 \tag{8}$$

$$u_j - u_i = F_j \qquad (i,j) \in A \tag{9}$$

*Proof:*
*Proving Eq.* (8) *(root node):*

$$F_0 \overset{\text{Def. 5}}{=} \theta_0 + \sum_{(0,i)\in A} f_{0i} \overset{\text{Def. 4}}{=} \theta_0 + \sum_{(0,i)\in A\setminus A_s} \tilde{\theta}_i \overset{\text{Def. 2}}{=} \tilde{\theta}_0 \overset{\text{Def. 3}}{=} u_0$$

*Proving Eq.* (9) *when* $(i,j) \in A \setminus A_s$

$$F_j \overset{\text{Def. 5}}{=} \theta_j + \sum_{(j,k)\in A} f_{jk} - f_{ij} \overset{\text{Def. 4}}{=} \theta_j + \sum_{(j,k)\in A\setminus A_s} \tilde{\theta}_k - \tilde{\theta}_j \overset{\text{Def. 2}}{=} \tilde{\theta}_j - \tilde{\theta}_j = 0 \overset{\text{Def. 3}}{=} u_j - u_i$$

*Proving Eq.* (9) *when* $(i,j) \in A_s$

$$F_j \overset{\text{Def. 5}}{=} \theta_j + \sum_{(j,k)\in A} f_{jk} - f_{ij} \overset{\text{Def. 4}}{=} \theta_j + \sum_{(j,k)\in A\setminus A_s} \tilde{\theta}_k \overset{\text{Def. 2}}{=} \tilde{\theta}_j \overset{\text{Def. 3}}{=} u_j - u_i$$

■

**Lemma 2.** *The interruption time $u_j$ is the sum of interruption residues on the path from the root to node $j$ (Eq. 10).*

$$u_j = \sum_{i\in path(0,j)} F_i \qquad\qquad i \in V \tag{10}$$

*Proof:* The proof will be made through induction on a path from the root to a node $j \in V$. First, it will be proven the base case, where node $j$ is the root ($j = 0$).

$$u_0 \overset{\text{Lem. 1}}{=} F_0 = \sum_{j\in path(0,0)} F_j$$

As induction hyphothesis, it will be assumed that Eq. (10) holds for any node in the path from the root to node $j$. Now it will be proven that Eq. (10) also holds for node $k$, such that $(j,k) \in A$. Starting with the identity given by Eq. (9):

$$u_k - u_j = F_k \quad \Rightarrow \quad u_k - \sum_{i\in path(0,j)} F_i = F_k \quad \Rightarrow \quad u_k = \sum_{i\in path(0,k)} F_i$$

■

Theo. 1 represents the core of the reliability evaluation methodology. It shows the ENS as an expression of the interruption time flows $f$, which means the ENS can be computed by solving a network flow problem.

**Theorem 1.** *The ENS can be expressed by interruption flow variables, according to Eq.* (11)*.*

$$ENS = \sum_{(i,j)\in A} (\tilde{l}_i - \tilde{l}_j) f_{ij} + \sum_{i\in V} \tilde{l}_i \theta_i \tag{11}$$

*Proof:*

$$\sum_{i\in V} l_i u_i \overset{\text{Lem. 2}}{=} \sum_{i\in V}\left(l_i \sum_{j\in path(0,i)} F_j\right) \overset{(a)}{=} \sum_{j\in V}\left(F_j \sum_{i\in V_j} l_i\right) \overset{\text{Eq. 1}}{=} \sum_{i\in V} \tilde{l}_i F_i$$
$$\overset{\text{Def. 5}}{=} \sum_{i\in V}\left(\tilde{l}_i\left(\theta_i + \sum_{(i,j)\in A} f_{ij}\right)\right) - \sum_{(i,j)\in A} \tilde{l}_j f_{ij} = \sum_{(i,j)\in A} (\tilde{l}_i - \tilde{l}_j) f_{ij} + \sum_{i\in V} \tilde{l}_i \theta_i$$

Equality $(a)$ is derived by noticing that $j \in path(0,i)$ if and only if $i \in V_j$. ∎

It is worth mentioning that other reliability indices, such as the SAIDI, can also be expressed in terms of variables $f$. This can be made by replacing the SAIDI interruption times $u$ from Eq. (2) with the interruption residues $F$, according to Eq. (10).

Lem. 3 shows how to determine the feasible ENS range, expressed by a lower and upperbound. The relative reliability state of a network can be established in terms of the distance from these bounds.

**Lemma 3.** *A lowerbound $E_{lb}$ and upperbound $E_{ub}$ for the ENS are described by the following inequality (12).*

$$\sum_{i\in V} \tilde{l}_i \theta_i = E_{lb} \leqslant ENS \leqslant E_{ub} = \tilde{l}_0 \sum_{i\in V} \theta_i \tag{12}$$

*Proof:* The lowerbound proof is trivial given that $E_{lb}$ is a constant present in Eq. (11). The scenario in which $ENS = E_{lb}$ implies $A_s = A$ and $f_{ij} = 0$ for all $(i,j) \in A$.

By inspecting Eq. (6), the interruption time flows $f$ are maximum if $A_s = \emptyset$. Using the assumption $A_s = \emptyset$, the maximum interruption time flows $f^{max}$ are given by Eq. (13).

$$f_{ij}^{max} \overset{\text{Def. 4}}{=} \tilde{\theta}_j \overset{\text{Def. 2}}{=} \theta_j + \sum_{(i,j)\in A\setminus A_s} \tilde{\theta}_j \overset{A_s=\emptyset}{=} \sum_{k\in V_j} \theta_k \qquad (i,j) \in A \tag{13}$$

The ENS upperbound $E_{ub}$ is derived by replacing the interruption flows $f_{ij}$ in Eq. (11) by the maximum interruption flows $f_{ij}^{max}$ of Eq. (13).

$$E_{ub} = \sum_{(i,j)\in A} (\tilde{l}_i - \tilde{l}_j) f_{ij}^{max} + \sum_{i\in V} \tilde{l}_i \theta_i = \sum_{(i,j)\in A}\left(\tilde{l}_i \sum_{k\in V_j} \theta_k\right) - \sum_{(i,j)\in A}\left(\tilde{l}_j \sum_{k\in V_j} \theta_k\right) + \sum_{i\in V} \tilde{l}_i \theta_i$$
$$= \sum_{i\in V}\left(\tilde{l}_i\left(\sum_{j\in V_i} \theta_j - \theta_i\right)\right) - \sum_{i\in V\setminus 0}\left(\tilde{l}_i \sum_{j\in V_i} \theta_j\right) + \sum_{i\in V} \tilde{l}_i \theta_i$$
$$= \tilde{l}_0\left(\sum_{i\in V} \theta_i - \theta_0\right) + \sum_{i\in V\setminus 0} \tilde{l}_i\left(\sum_{j\in V_i} \theta_j - \theta_i\right) - \sum_{i\in V\setminus 0}\left(\tilde{l}_i \sum_{j\in V_i} \theta_j\right) + \sum_{i\in V} \tilde{l}_i \theta_i$$
$$= \tilde{l}_0 \sum_{i\in V} \theta_i - \tilde{l}_0 \theta_0 - \sum_{i\in V\setminus 0} \tilde{l}_i \theta_i + \sum_{i\in V} \tilde{l}_i \theta_i = \tilde{l}_0 \sum_{i\in V} \theta_i$$

∎

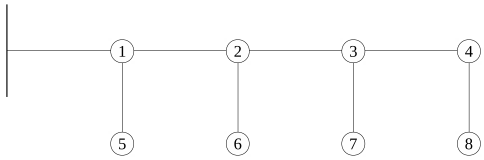
## 5 Illustrative numerical examples

The networks shown in Figure 2 are used as application examples of the methodology. These networks were extracted from Billinton and Allan (1996) and they all contain eight nodes from which four are load points (nodes $5-8$). It is considered that the substation (root) has zero failure rate, and this is why node 0 was ommited. Two scenarios are tested to measure the effect of automatic switches on the network reliability. All parameters used in the tests were taken from Billinton and Allan (1996) and replicated in Table 1 and the evaluation results are shown in Table 2. The ENS was computed to both scenarios through Eq. (11), which for these networks results in the following expression:

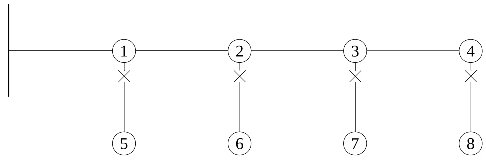
$$\begin{aligned} ENS =& (\tilde{l}_1 - \tilde{l}_2) f_{12} + (\tilde{l}_1 - \tilde{l}_5) f_{15} + (\tilde{l}_2 - \tilde{l}_3) f_{23} + (\tilde{l}_2 - \tilde{l}_6) f_{26} + (\tilde{l}_3 - \tilde{l}_4) f_{34} \\ &+ (\tilde{l}_3 - \tilde{l}_7) f_{37} + (\tilde{l}_4 - \tilde{l}_8) f_{48} + \tilde{l}_0\theta_0 + \tilde{l}_1\theta_1 + \tilde{l}_2\theta_2 + \tilde{l}_3\theta_3 + \tilde{l}_4\theta_4 + \tilde{l}_5\theta_5 + \tilde{l}_6\theta_6 + \tilde{l}_7\theta_7 + \tilde{l}_8\theta_8 \end{aligned}$$

*Scenario 1 – clear network:* The network is completely clear, without a single automatic switch (Figure 2(a)). Every fault will cause the substation main breaker to operate, disconnecting the entire network. The network ENS in this scenario is equal to $84,000\ kWh/year$.

*Scenario 2 – lateral distribution switches:* The network contains switches on each predecessor arc of a load point (Figure 2(b)). A fault on a load point causes its corresponding switch to operate so that no other node is affected. This explains the interruption time flows equal to zero on arcs $(1,5)$, $(2,6)$, $(3,7)$ and $(4,8)$. The network ENS in this scenario is equal to $54,800\ kWh/year$.



(a) Scenario 1: clear network



(b) Scenario 2: lateral distribution switches

**Fig. 2.** Scenarios tested in the numerical examples.

**Table 1.** Parameters used in the numerical examples.

| | | Node $i$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $l_i$ | $kW$ | 0 | 0 | 0 | 0 | 5000 | 4000 | 3000 | 2000 |
| $\tilde{l}_i$ | $kW$ | 14000 | 9000 | 5000 | 2000 | 5000 | 4000 | 3000 | 2000 |
| $\lambda_i$ | $\varphi/y$ | 0.2 | 0.1 | 0.3 | 0.2 | 0.2 | 0.6 | 0.4 | 0.2 |
| $t_i$ | $h/\varphi$ | 4.0 | 4.0 | 4.0 | 4.0 | 2.0 | 2.0 | 2.0 | 2.0 |
| $\theta_i$ | $h/y$ | 0.8 | 0.8 | 1.2 | 0.8 | 0.4 | 1.2 | 0.8 | 0.4 |

$\varphi$ – failure, $h$ – hour, $kW$ – kilowatt, $y$ – year.
$l_i$, $\tilde{l}_i$ – power load and downstream power load of node $i$.
$\lambda_i$ – node $i$ failure rate.
$t_i$ – average interruption time.
$\theta_i$ – node $i$ self-interruption time.

**Table 2.** Reliability evaluation for the numerical examples.

Scenario 1 – clear network

| | Node $i$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $\tilde{\theta}_i\ h/y$ | 6 | 4.8 | 3.2 | 1.2 | 0.4 | 1.2 | 0.8 | 0.4 |
| $F_i\ h/y$ | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_i\ h/y$ | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |

| | Arc $(i,j)$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | $(1,2)$ | $(1,5)$ | $(2,3)$ | $(2,6)$ | $(3,4)$ | $(3,7)$ | $(4,8)$ |
| $f_{ij}\ h/y$ | 4.8 | 0.4 | 3.2 | 1.2 | 1.2 | 0.8 | 0.4 |

Scenario 2 - lateral distribution switches

| | Node $i$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $\tilde{\theta}_i\ h/y$ | 3.2 | 2.4 | 2 | 0.8 | 0.4 | 1.2 | 0.8 | 0.40 |
| $F_i\ h/y$ | 3.2 | 0 | 0 | 0 | 0.4 | 1.2 | 0.8 | 0.40 |
| $u_i\ h/y$ | 3.2 | 3.2 | 3.2 | 3.2 | 3.6 | 4.4 | 4 | 3.60 |

| | Arc $(i,j)$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | $(1,2)$ | $(1,5)$ | $(2,3)$ | $(2,6)$ | $(3,4)$ | $(3,7)$ | $(4,8)$ |
| $f_{ij}\ h/y$ | 2.4 | 0 | 2 | 0 | 0.8 | 0 | 0 |

$h$ – hour, $y$ – year
$\tilde{\theta}_i$ – downstream interruption time of node $i$.
$F_i$ – interruption residue of node $i$.
$u_i$ – interruption time of node $i$.
$f_{ij}$ – interruption time flow on arc $(i,j)$.

## 6 MILP model

This section shows how the proposed methodology can be used to formulate the switch allocation problem as a mixed integer linear programming model. Different criteria can be adopted when solving the switch allocation problem. For example, minimizing the ENS, SAIDI, allocation costs or a combination of these, are equally interesting investigations. In this paper, it is considered a fixed number of $N$ switches that must be allocated on the best possible locations in the network so as to minimize the ENS.

In the following MILP model (SAP), a switch is allocated on arc $(i,j)$ if and only if $x_{ij}=1$. Variable $f_{ij}$ gives the interruption time flow (Def. 4) on arc $(i,j)$. Parameters $\tilde{l}$, $\theta_i$ and $E_{lb}$ are described by Eqs. (1,3,12). A constant $M_i$ is defined for each node $i$ as $M_i = \sum_{j\in V_i}\theta_j$ and its importance is explained in Lem. 4.

(SAP)

$$MIN \quad \sum_{(i,j)\in A} (\tilde{l}_i - \tilde{l}_j) f_{ij} + E_{lb} \tag{14}$$

s.t.

*(number of switches)*

$$\sum_{(i,j)\in A} x_{ij} \leqslant N \tag{15}$$

*(interruption time flows)*

$$f_{ij} - \sum_{(j,k)\in A} f_{jk} + M_j x_{ij} \geqslant \theta_j \qquad (i,j)\in A \tag{16}$$

*(variables bounds and integrality)*

$$f_{ij} \geqslant 0 \qquad (i,j)\in A \tag{17}$$

$$x_{ij} \in \{0,1\} \qquad (i,j)\in A \tag{18}$$

The objective function (14) represents the solution ENS. The number of switches is constrained by (15). The interruption time flows are given by (16). The model correctness relies on the accuracy of the interruption time flows. Lem. 4 shows that the flow values $f^*_{ij}$ of an optimal solution match Def. 4.

**Lemma 4.** *There is an optimal solution $(x^*,f^*)$ for formulation (SAP) such that, for $A_s = \{(i,j)\in A : x^*_{ij}=1\}$, we have that Eq. (19) holds.*

$$f^*_{ij} = \begin{cases} 0 & (i,j)\in A_s \\ \tilde{\theta}_j & (i,j)\in A\setminus A_s \end{cases} \tag{19}$$

*Proof:* First we shall prove inequality (20) through induction, then we argue that there exist an optimal solution which satisfies (20) in equality.

$$f^*_{ij} \geqslant \begin{cases} 0 & (i,j)\in A_s \\ \tilde{\theta}_j & (i,j)\in A\setminus A_s \end{cases} \tag{20}$$

*Step 1.* Proving base case in which node $j$ is a leaf. Starting from (16):

$$f^*_{ij} + M_j x^*_{ij} \geqslant \theta_j$$

Case $(i,j)\in A_s$. This case implies $x^*_{ij}=1$, thus:

$$f^*_{ij} + M_j = f^*_{ij} + \theta_j \geqslant \theta_j \Rightarrow f^*_{ij} \geqslant 0$$

Case $(i,j) \in A \setminus A_s$. This case implies $x^*_{ij} = 0$, which leads to:

$$f^*_{ij} \geqslant \theta_j = \tilde{\theta}_j$$

*Step 2.* Assuming as induction hypothesis that inequality (20) is true for arcs downstream of $j$.

*Step 3.* Proving that (20) is true for arc $(i,j)$. Starting from (16):

$$f^*_{ij} - \sum_{(j,k)\in A} f^*_{jk} + M_j x^*_{ij} \geqslant \theta_j$$

Case $(i,j) \in A_s$. This case implies $x^*_{ij} = 1$, thus:

$$\begin{aligned}
& f^*_{ij} - \sum_{(j,k)\in A} f^*_{jk} + M_j \geqslant \theta_j \\
& \Rightarrow f^*_{ij} + M_j \geqslant \theta_j + \sum_{(j,k)\in A} f^*_{jk} \geqslant \theta_j + \sum_{(j,k)\in A\setminus A_S} \tilde{\theta}_k = \tilde{\theta}_j \\
& \Rightarrow f^*_{ij} + \sum_{k\in V_j} \theta_k \geqslant f^*_{ij} + \tilde{\theta}_j \geqslant \tilde{\theta}_j \\
& \Rightarrow f^*_{ij} \geqslant 0
\end{aligned}$$

Case $(i,j) \in A \setminus A_s$. This case implies $x^*_{ij} = 0$. Taking away $M_j$ from the previous case will lead to:

$$f^*_{ij} \geqslant \tilde{\theta}_j$$

*Step 4.* Proving that there is an optimal solution that holds (20) in equality:

The switch allocation problem implies the minimization of the interruption time flows (14). Therefore, an optimal flow value $f^*_{ij}$ should hold (20) in equality, unless its coefficient in the objective function $(\tilde{l}_i - \tilde{l}_j)$ is zero. ■

## 7 Branch-and-cut framework

Two sets of valid inequalities are proposed to strengthen the MILP model for the switch allocation problem. They are the *interruption time flow* inequalities and *extended interruption time flow* inequalities.

### 7.1 Interruption time flow inequalities

The interruption time flow inequalities relies on the fact that the interruption time flow $f_{ij}$ should be at least equal to node $j$ self-interruption time $\theta_j$ when there is no switch on arc $(i,j)$. This is expressed by constraints (21), and since there are only $|A|$ of such constraints, all of them can be inserted directly in the model.

$$f_{ij} \geqslant \theta_j(1 - x_{ij}) \qquad (i,j) \in A \tag{21}$$

### 7.2 Extended interruption time flow inequalities

The interruption time flow inequalities can be extended to a much stronger set of valid inequalities that takes into consideration the self-interruption times from all nodes downstream of node $j$. For instance, consider there are only two nodes downstream of $j$: $j$ itself and $k$ ($V_j = \{j,k\}$). In this case, the interruption time flow $f_{ij}$ could assume only three values; $f_{ij} = 0$ if there is a switch on arc $(i,j)$; $f_{ij} = \theta_j$ if there is a switch on arc $(j,k)$ but no switch on arc $(i,j)$; $f_{ij} = \theta_j + \theta_k$ if there is a switch on neither arcs $(i,j)$ nor $(j,k)$. Under these considerations, the following inequality is valid:

$$f_{ij} \geqslant \theta_j(1 - x_{ij}) + \theta_k(1 - x_{ij} - x_{jk})$$

By extending these ideas to any possible subset $S \in V_j$ of downstream nodes of $j$, the set of valid inequalities (22) can be derived.

$$f_{ij} \geqslant \sum_{k \in S} \theta_k (1 - \tilde{x}_{ik}) \qquad (i,j) \in A, S \subseteq V_j \tag{22}$$

Where

$$\tilde{x}_{ik} = \sum_{(s,t) \in path(i,k)} x_{st} \qquad (i,j) \in A, k \in V_j$$

There is an exponential number of constraints (22). However, the most violated of these constraints for each arc $(i,j)$ can be found by solving the following optimization problem:

$$\phi_{ij} = \max_{S \in V_j} \left[ \sum_{k \in S} \theta_k (1 - \tilde{x}_{ik}) \right] \qquad (i,j) \in A \tag{23}$$

To solve problem (23) in polynomial time it suffices to perform a depth first search on the subtree induced by $V_j$. If a node $k \in V_j$ results in $(1 - \tilde{x}_{ik}) > 0$ then $k$ is included on set $S$ and the search continues downstream; otherwise, the search backtracks upstream from node $k$. If $f_{ij} < \phi_{ij}$, then a violated constraint has been found.

## 8 Computational studies

The switch allocation problem was solved for a set of benchmark networks (Kavasseri and Ababei 2013), whose attributes are described in Table 3. The SAP model (14,15,16,18,21), described in Sec. 6, was embedded with the extended interruption time flow constraints (22), within a branch-and-cut framework. This extended model is referred as SAP-E. The SAP and SAP-E models were loaded in Gurobi 5.5 and solved with a time limit of ten minutes on an Intel i7 3930k with 16 GB of RAM, and Ubuntu 12.04 as operating system.

Each network was solved adopting fixed-values of the maximum percentage of switches in a solution against the number of candidate arcs ($P = [20\%, 40\%, 60\%, 80\%]$). In other words, the maximum number of switches in a solution is given by $N = \lfloor P \cdot |A| \rfloor$. Five networks and four values of $P$ resulted in a total of 20 instances. The objective function (ENS) of the best feasible solutions ($UB$) and the execution times ($CPU$) are shown in Table 4 for models SAP and SAP-E.

The extended model clearly outperforms SAP in both solution quality and execution time. The SAP-E found optimal solutions for all 20 instances within very small execution times (0.17 second on average and 0.99 second in the worst case). The SAP was able to prove the optimum for only 11 instances with a GAP of 0.91% and 270.08 seconds of execution time, on average. The hardest instance for the SAP was the R7 network with $P = 20\%$, for which a 8.21% GAP was attained. Nevertheless, the SAP does have a good performance on practice, since it reached the optimum for 17 instances and found close to optimal solutions for the three remaining instances. This shows that the performance of the SAP-E relies on the strength of the extended interruption time flow constraints to cut fractional solutions. This unburdens the MILP solver from the process of proving optimality, thus speeding the algorithm.

**Table 3.** Benchmark networks attributes.

| network | feeders | $\|V\|$ | $\|A\|$ | $\tilde{l}_0$ | $E_{lb}$ | $E_{ub}$ |
|---|---|---|---|---|---|---|
| R3 | 1 | 34 | 33 | 3,708.27 | 2,069.97 | 11,135.23 |
| R4 | 11 | 95 | 94 | 28,342.96 | 2,340.32 | 4,242.33 |
| R5 | 8 | 144 | 143 | 18,315.82 | 3,747.42 | 14,110.97 |
| R6 | 3 | 205 | 204 | 27,571.37 | 1,437.63 | 6,932.57 |
| R7 | 7 | 881 | 880 | 124,920.01 | 266,293.63 | 1,518,308.94 |

$E_{lb}$ – ENS lowerbound ($kWh$/year) Eq. (12)
$E_{ub}$ – ENS upperbound ($kWh$/year) Eq. (12)
$\tilde{l}_0$ – total power load ($kW$) Eq. (1)

**Table 4.** Computational results summary.

$P = 20\%$

| | | SAP | | | SAP-E | | |
|---|---|---|---|---|---|---|---|
| network | $N$ | $UB$ | $GAP(\%)$ | $CPU$ | $UB$ | $GAP(\%)$ | $CPU$ |
| R3 | 6 | **2715.24** | **opt** | < 0.01 | **2715.24** | **opt** | < 0.01 |
| R4 | 18 | **2504.72** | **opt** | 0.14 | **2504.72** | **opt** | < 0.01 |
| R5 | 28 | **4801.43** | **opt** | 0.63 | **4801.43** | **opt** | 0.02 |
| R6 | 40 | **1661.86** | 2.24 | 600.00 | **1661.86** | **opt** | 0.08 |
| R7 | 176 | 307668.14 | 8.21 | 600.00 | **307092.99** | **opt** | 0.54 |

$P = 40\%$

| | | SAP | | | SAP-E | | |
|---|---|---|---|---|---|---|---|
| network | $N$ | $UB$ | $GAP(\%)$ | $CPU$ | $UB$ | $GAP(\%)$ | $CPU$ |
| R3 | 13 | **2269.21** | **opt** | 0.23 | **2269.21** | **opt** | < 0.01 |
| R4 | 37 | **2361.50** | **opt** | 0.27 | **2361.50** | **opt** | < 0.01 |
| R5 | 57 | **3928.03** | 2.47 | 600.00 | **3928.03** | **opt** | 0.02 |
| R6 | 81 | **1457.86** | 1.11 | 600.00 | **1457.86** | **opt** | 0.60 |
| R7 | 352 | 274710.83 | 2.92 | 600.00 | **274693.02** | **opt** | 0.55 |

$P = 60\%$

| | | SAP | | | SAP-E | | |
|---|---|---|---|---|---|---|---|
| network | $N$ | $UB$ | $GAP(\%)$ | $CPU$ | $UB$ | $GAP(\%)$ | $CPU$ |
| R3 | 19 | **2144.88** | **opt** | 0.36 | **2144.88** | **opt** | < 0.01 |
| R4 | 56 | **2340.32** | **opt** | < 0.01 | **2340.32** | **opt** | < 0.01 |
| R5 | 85 | **3774.88** | 0.31 | 600.00 | **3774.88** | **opt** | 0.02 |
| R6 | 122 | **1439.19** | 0.05 | 600.00 | **1439.19** | **opt** | 0.99 |
| R7 | 528 | **268135.66** | 0.69 | 600.00 | **268135.66** | **opt** | 0.28 |

$P = 80\%$

| | | SAP | | | SAP-E | | |
|---|---|---|---|---|---|---|---|
| network | $N$ | $UB$ | $GAP(\%)$ | $CPU$ | $UB$ | $GAP(\%)$ | $CPU$ |
| R3 | 26 | **2089.06** | **opt** | 0.04 | **2089.06** | **opt** | < 0.01 |
| R4 | 75 | **2340.32** | **opt** | < 0.01 | **2340.32** | **opt** | < 0.01 |
| R5 | 114 | **3747.42** | **opt** | < 0.01 | **3747.42** | **opt** | < 0.01 |
| R6 | 163 | **1437.63** | **opt** | < 0.01 | **1437.63** | **opt** | < 0.01 |
| R7 | 704 | 266548.57 | 0.09 | 600.00 | **266547.33** | **opt** | 0.28 |

$N = \lfloor P * |A| \rfloor$ – maximum number of switches in the solution.
$P$ – percentage of switches against number of arcs.
SAP – mixed integer linear model for the switch allocation problem.
SAP-E – SAP extended with the interruption time flow constraints.
$UB$ – ENS ($kWh$/year) of the best feasible solution found.
$CPU$ – execution time (seconds).
$GAP(\%) = 100 \cdot (UB - LB)/UB$

## 9 Final remarks

This paper proposes a mixed integer linear model for a switch allocation problem, in which a fixed number of switches must be installed in a power distribution network to reduce the energy not supplied. A set of valid inequalities are derived to cut fractional solutions and speed the solution process. Case studies were performed using a benchmark set of distribution networks to evaluate the model and the strength of the valid inequalities. The results show that the branch-and-cut methodology proved optimality for all instances within a few seconds of computational time. This sets a new benchmark for solving the switch allocation problem, with significant implications on the evaluation and improvement of power distribution reliability.

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