A MUTUAL ACQUAINTANCE PARTY PROBLEM

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RESUMO

Problemas de festa deram origem a teoremas que são parte do fundamento da teoria de grafos. Consideramos aqui um problema de festa de amigos em comum cuja solução pode ser representada por grafos onde cada par de vértices tem um vizinho comum. Tais grafos são exatamente aqueles de diâmetro 2 com toda aresta em um triângulo. Nessa classe de grafos procuramos aqueles com a maior ordem relativamente ao seu grau máximo, isto é, estudamos o problema do grau-diâmetro restrito a essa classe de grafos. Denotando por Δ o grau máximo de um grafo G, obtemos uma nova cota superior para a ordem de tais grafos: para $\Delta \geq 3$ ímpar, temos $|V(G)| \leq \Delta(\Delta - 1) - 2$, melhorando o resultado anterior de Hahn et. al (2002). Construímos uma família infinita de tais grafos com ordem $|V(G)| \geq \left(\frac{\Delta+3}{2}\right)^2 - 1$. Modificando essa construção obtemos um grafo G de ordem $|V(G)| = (\Delta - 1)^2$. Como aplicação dos nossos resultados solucionamos o problema de festa de amigos em comum considerado aqui para $\Delta = 3$, ficamos muito próximos de resolvê-lo para $\Delta = 5$ e estabelecemos cotas superiores e inferiores para uma infinidade de valores de Δ.

PALAVRAS-CHAVE: Problema do grau-diâmetro; Grafos de diâmetro dois; Grafos fortemente regulares.

ABSTRACT

Party problems have yielded foundational theorems of graph theory in its early days. Here we consider a mutual acquaintance party problem whose solutions can be represented by graphs where each pair of its vertices has a common neighbor. Such graphs are exactly those of diameter two with every edge in a triangle. In this class of graphs we search for those with the largest order relatively to their maximum degree, that is, we study the degree-diameter problem restricted to such class of graphs. Denoting by Δ the maximum degree of a graph G, we obtain a new upper bound on the order of such graphs: for odd $\Delta \geq 3$ we have $|V(G)| \leq \Delta(\Delta - 1) - 2$, improving the result previously obtained by Hahn et. al (2002). We construct an infinite family of such graphs with order $|V(G)| \geq \left(\frac{\Delta+3}{2}\right)^2 - 1$. Modifying this construction we obtain such a graph G of order $|V(G)| = (\Delta - 1)^2$. As an application of our methods and results we solve the considered mutual acquaintance problem for $\Delta = 3$, we get very close to solve for $\Delta = 5$ and we establish upper and lower bounds for infinitely many values of Δ.

KEYWORDS: Degree-diameter problem; Diameter two graphs; Strongly regular graphs.

Main Area: Theory and Algorithms in Graphs.

1 Introduction

In this work we are concerned with finite, simple, undirected graphs. For a graph G, we use $V(G)$, $E(G)$, $|V(G)|$, and $\Delta(G)$ to denote its vertex set, edge set, order, and maximum degree, respectively. We call two vertices connected by an edge, *adjacent vertices* or *neighbors*. The *degree* of a vertex is the number of incident edges to that vertex or, equivalently, the amount of neighbors it has. We denote by $N_G(v)$ the set of neighbors of a vertex v in G. For $v \in V(G)$, let $deg_G(v)$ denote the degree of v in G. If there is no confusion about the context, we write Δ for $\Delta(G)$, $deg(v)$ for $deg_G(v)$ respectively. A *path* of lenght i from v_0 to v_i in a graph is a sequence of $i + 1$ distinct vertices starting with v_0 and ending with v_i such that consecutive vertices are adjacent. The *distance* between two vertices is the length of a shortest path between them. The *diameter* of a graph is the maximum distance over all pairs of vertices. A graph is called *complete* if every pair of its vertices are adjacent. The complete graph on n vertices is denoted by K_n.

A party yields a graph G if we let the people at the party be represented by vertices and then let two vertices be connected by an edge if those two people are acquainted. Party problems trace their roots back to the relatively early days of graph theory. We recall here two remarkable examples of theorems obtained as solutions to party problems. The first example is known as the Theorem on Friends and Strangers. In layman terms:

Theorem on Friends and Strangers: *"In any party of six people, either at least three of them are pairwise strangers or at least three of them are pairwise acquaintances".*

This seemingly simple fact is what motivated Ramsey's theorem, which is the foundation of an entire research area, known nowadays as Ramsey Theory.

Another theorem obtained from a party problem is the so-called Friendship Theorem, which can be stated in layman terms as follows:

Friendship Theorem: *"If we have a party of people where any two people have exactly one friend in common, then there is a person who is everybody's friend".*

Most authors recognize that Erdös, Rényi and Sós in 1966 published the first proof of this theorem. A stronger statement is actually true: the only graph with this structure consists of a collection of triangles that all share one vertex. This theorem was first proved in connection with the problem of minimizing the edge set of a graph of fixed maximum degree and smallest maximum distance. Since then many different proofs and some variations of this theorem have been given by other authors.

The friendship theorem can be rephrased in this way: in a graph G with n vertices, if every two vertices have exactly one common neighbor in G, then there exists a vertex v in G with maximum degree , that is, $deg(v) = n-1$. A graph G with this structure has diameter 2, maximum degree $n-1$ and n vertices. Therefore the order of G is not large relatively to its maximum degree.

When we relax the hypothesis of the friendship theorem by considering a party where any two people have at least one friend in common, we must consider graphs where any pair of its vertices has at least one common neighbor. Those graphs are exactly the graphs of diameter 2 with the property that every edge is in a triangle and the complete graphs K_n. From this weaker hypothesis we consider the following mutual acquaintance party problem:

Mutual Acquaintance Party Problem: *"How many people can we have in a party where any two people have a mutual acquaintance and each person has at most Δ acquaintances? "*

To solve this problem we must find, among graphs with diameter two and every edge in a triangle, those with the largest order relatively to their maximum degree.

The study of large graphs of given maximum degree and diameter is known as the degree-diameter problem and has a rich history (see the survey by Miller and Siran, (2005)). The degree-diameter problem was also motivated by questions in the design of interconnection networks in theoretical computer science and has often been restricted to special classes of graphs like vertex-transitive and Cayley graphs (for the definition of those classes of graphs see Godsil and Royle

(2001), pages 33-34). Looking for large graphs of diameter 2 with every edge in a triangle can be considered a restricted version of the degree-diameter problem.

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same colour.

The common neighborhood graph of G, denoted here by $con(G)$, is the graph defined by:

$$V(con(G)) = V(G) \text{ and } E(con(G)) = \{uv \mid u \text{ and } v \text{ have a common neighbor in } G\}.$$

The *injective chromatic number* of a graph G, denoted by $\chi_i(G)$, can be defined as the chromatic number of $con(G)$:

$$\chi_i(G) = \chi(con(G)) \leq |V(con(G))| = |V(G)|.$$

By the definition above, the graphs where each pair of its vertices has a common neighbor must have maximum injective chromatic number: $\chi_i(G) = |V(G)|$. It has been proved (Hahn *et. al* (2002)) that, if G has maximum injective chromatic number then $|V(G)| \leq \Delta(\Delta - 1)$, for $\Delta \geq 3$. One of our results reduces this general upper bound to $\Delta(\Delta - 1) - 2$, for odd $\Delta \geq 3$.

A *strongly regular graph* with parameters (n, k, λ, μ) is a graph G of order n, $(n \geq 3)$ wich is regular of degree k and satisfies the following additional requirements (Brualdi and Ryser (1991)):

(i) any pair of adjacent vertices has λ common neighbours;

(ii) any pair of non-adjacent vertices has μ common neighbours.

Strongly regular graphs (introduced by Bose, (1963)) with nonzero parameters (n, k, λ, μ) have diameter two and also every edge in a triangle and therefore belong to the class we are interested in here. The previously largest known infinite family of graphs of diameter two with every edge in a triangle were the Cartesian products $K_r \square K_r$ (a.k.a. square lattice graphs) which are strongly regular of order $n = \left(\frac{k+2}{2}\right)^2$ (see Elzinga, (2003)). In one of our results we construct an infinite family of graphs of diameter two with every edge in a triangle with order $|V(G)| \geq \left(\frac{\Delta+3}{2}\right)^2 - 1$.

The largest possible strongly regular graphs with nonzero parameters could have order $n = (k-1)^2$ but it is still unknown if such a graph exists. The largest known strongly regular graphs with nonzero parameters have order $n = \frac{k^2+2}{2}$ and there exists at most five of them. We construct a graph of order $|V(G)| = (\Delta - 1)^2 > \frac{\Delta^2+2}{2}$ of diameter two with every edge in a triangle.

2 Attacking the problem from below

Let r be an integer, $r \geq 5$. Consider the Cartesian product of two complete graphs on $r - 1$ and r vertices respectively, denoted by $K_{r-1} \square K_r$ (also known as the $(r - 1) \times r$ lattice graph). We can think of its vertices arranged in a rectangular grid with $r - 1$ rows and r columns where each row is a r-clique and each column is a $r - 1$-clique. We have that $K_{r-1} \square K_r$ are regular graphs with degree $\Delta(K_{r-1} \square K_r) = 2r - 3$ and order $|V(K_{r-1} \square K_r)| = (r - 1)r = \frac{\Delta+1}{2}\frac{\Delta+3}{2} = \left(\frac{\Delta+2}{2}\right)^2 - \frac{1}{4}$. We may assign integer coordinates (i, j) to the vertices, where $1 \leq i \leq r - 1$ and $1 \leq j \leq r$. Two vertices are adjacent if and only if they belong to the same row or the same column; that is, (i_1, j_1) and (i_2, j_2) are adjacent if and only if either $i_1 = i_2$ or $j_1 = j_2$.

Given any pair of non-adjacent vertices in $K_{r-1} \square K_r$ denoted by (i_1, j_1) and (i_2, j_2), they have exactly two common neighbors: (i_1, j_2) and (i_2, j_1). Thus, given any pair of vertices in $K_{r-1} \square K_r$ we know that they have at least two common neighbors. Doing the same analysis for the cartesian products $K_{r-2} \square K_r$ we can see that they have degree $\Delta(K_{r-2} \square K_r) = 2r - 4$, order $|V(K_{r-2} \square K_r)| = (r - 2)r = \frac{\Delta}{2}\frac{\Delta+4}{2} = \left(\frac{\Delta+2}{2}\right)^2 - 1$ and, for $r \geq 6$, any pair of vertices in $K_{r-2} \square K_r$ have at least two common neighbors.

Consider now $r = 2p \geq 6$. Starting with $K_{r-1} \square K_r$, we will remove $(r-1)\frac{r}{2}$ edges, never removing two edges from the same vertex. This will leave each vertex with exactly one less neighbor, and therefore the graph G obtained will have degree $\Delta(G) = 2r - 4$ and order $|V(G)| = (r - 1)r = \frac{\Delta+2}{2}\frac{\Delta+4}{2} = \left(\frac{\Delta+3}{2}\right)^2 - \frac{1}{4}$. Doing the same kind of edge removal but starting from $K_{r-2} \square K_r$ the graph H obtained will have degree $\Delta(H) = 2r - 5 =$ and order $|V(H)| = (r - 2)r = \frac{\Delta+1}{2}\frac{\Delta+5}{2} = \left(\frac{\Delta+3}{2}\right)^2 - 1$.

We will remove only “horizontal" edges, that is, edges connecting vertices on the same row. For each row of vertices in $K_{r-1}\Box K_r$ we will label the horizontal edges by the columns they link: that is, the edge $[(1,1)(1,2)]$ will be refered as edge 12 of row 1. As an example, for $r = 6$ we could select the following edges from $K_5\Box K_6$:

$$\begin{array}{ccc} 12 & 35 & 46 \\ 13 & 26 & 45 \\ 14 & 23 & 56 \\ 15 & 24 & 36 \\ 16 & 25 & 34 \end{array}. \tag{2.1}$$

At the end of the edge removal we want each vertex with exactly one less neighbor and we wish to remove only one horizontal edge for each pair of columns (because when we remove the horizontal edge $j_1 j_2$ from row i_3 we will have that $v_{i_4 j_2}$ is now the only common neighbor of $v_{i_3 j_2}$ and $v_{i_4 j_1}$). If we can do that, starting from $K_{r-1}\Box K_r$, we have $\frac{r(r-1)}{2}$ pairs of columns. Our conditions for edge removal plus this labeling we chose, makes the selection of the horizontal edges to be removed equivalent to arrange the $\frac{r(r-1)}{2}$ pairs,

$$12,\ldots,1r\,,23,\ \ldots,2r\,,\ldots,(r-1)r\,,$$

into $r-1$ lines, $\frac{r}{2}$ pairs in each row, in a way that pairs on the same row have no digit in common. (this guarantees that we are not removing two edges from the same vertex). We have above, in (2.1), such an example. This is equivalent to arrange the pairs $\{12,\ldots,(r-1)r\}$ in $r-1$ rows with each row being a permutation of $\{1,\ 2,\ \ldots,\ r\}$.

Remark 2.1. *We are not aware of any proof that, the pairs* $\{12,\ldots,1r,23,\ldots,2r,\ldots,(r-1)r\}$ *can be arranged in* $r-1$ *rows with each row being a permutation of* $\{1,\ 2,\ \ldots,\ r\}$ *for every* $r = 2p$.

In the Lemma 2.2 we provide a way to arrange the pairs $\{12,\ldots,1r,23,\ldots,2r,\ldots,(r-1)r\}$ in $r-1$ rows with each row being a permutation of $\{1,\ 2,\ \ldots,\ r\}$ for every $r = 2^{q+1}$.

Lemma 2.2. *For* $r = 2^{q+1}$ *we can arrange the pairs* $\{12,\ldots,1r,23,\ldots,2r,\ldots,(r-1)r\}$ *in* $r-1$ *rows with each row being a permutation of* $\{1,\ 2,\ \ldots,\ r\}$.

Proof. The first row will be just the pairs $12, 34,\ldots,(r-1)r$. If $q = 0$ we are done.

For $q \geq 1$, from the set $\{1,\ldots,\ r\}$, we take its first half, that is, $\{1,\ldots,\ \frac{r}{2}\}$ and use each of those symbols, in that same order (crescent order), as the first "digit" of a pair. Then we take the second half, that is, $\{\frac{r}{2}+1,\ldots,\ r\}$ and use each of those symbols, in that same order, as the second "digit" of each one of the $\frac{r}{2}$ pairs. Keeping fixed the first digit of each pair and doing cyclic permutations on the second digits of those pairs, we will get $\frac{r}{2}$ different rows, without repeating any pair, and each row will be a permutation of $\{1,\ 2,\ \ldots,\ r\}$. If $q = 1$ we are done.

For $q \geq 2$ we take the first half of $\{1,\ 2,\ \ldots,\ \frac{r}{2}\}$, that is, $\{1,\ 2,\ \ldots,\ \frac{r}{4}\}$ and use each of those symbols, in crescent order, as the first "digit" of a pair. Then we take the second half, that is, $\{\frac{r}{4}+1,\ldots,\ \frac{r}{2}\}$ and use each of those symbols, in that same order, as the second "digit" of each one of the $\frac{r}{4}$ pairs. To generate $\frac{r}{4}$ more pairs, take the first half of $\{\frac{r}{2}+1,\ldots,\ r\}$, that is , $\{\frac{r}{2}+1,\ldots,\ \frac{3r}{4}\}$ and use each of those symbols, in crescent order, as the first "digit" of a pair, then we take the second half, that is, $\{\frac{3r}{4}+1,\ldots,\ r\}$ and use each of those symbols, in that same order, as the second "digit" of each one of the pairs, obtaining $\frac{r}{4}$ more. Then we have one row, composed of $\frac{r}{2} = \frac{r}{4} + \frac{r}{4}$ pairs. Each time we do a cyclic permutation on the second "digits" of the first $\frac{r}{4}$ pairs and a cyclic permutation on the second "digits" of the other $\frac{r}{4}$ pairs we get a new row. In this way we will get $\frac{r}{4}$ different rows, without repeating any pair, and each row will be a permutation of $\{1,\ 2,\ \ldots,\ r\}$. If $q = 2$ we are done.

As $r = 2^{q+1}$, this halving proccess can be repeated a total of q times. In the ℓ-th step the cyclic permutations yield $\frac{r}{2^\ell}$ rows. So with the default first row plus the ones generated by this proccess we will have the following total number of rows:

$$1 + \sum_{\ell=1}^{q} \frac{r}{2^\ell} = 1 + r\left(1 - \frac{1}{2^q}\right) = 1 + r - 2 = r - 1. \tag{2.2}$$

□

We can now construct infinite families of large graphs of diameter two with every edge in a triangle.

Theorem 2.3. *For infinitely many even values of $\Delta \geq 8$ there is a graph G of diameter two with every edge in a triangle having order $|V(G)| = \left(\frac{\Delta+3}{2}\right)^2 - \frac{1}{4}$. Also, for infinitely many odd values of $\Delta \geq 7$, there is such a graph H of order $|V(H)| = \left(\frac{\Delta+3}{2}\right)^2 - 1$.*

Proof. Consider $r = 2^{q+1} \geq 8$. We start with $K_{r-1}\square K_r$, where $\Delta(K_{r-1}\square K_r) = 2r - 3$. By lemma 2.2 there is an arrangement of the pairs $\{12, \dots, 1r, 23, \dots, 2r, \dots, (r-1)r\}$ in $r - 1$ rows with each row being a permutation of $\{1,\ 2,\ \dots,\ r\}$. We will use such an arrangement to select the "horizontal" edges to be removed from $K_{r-1}\square K_r$ to obtain then a new graph G with the same vertex set, that is $V(G) = V(K_{r-1}\square K_r)$. Such a selection has the property that, after the removal, each vertex has exactly one less neighbor and pairs of vertices that had two common neighbors in $K_{r-1}\square K_r$ will have at least one common neighbor in G. After this procedure of edge removal, the amount of common neighbors for pairs of vertices on the same column will remain $r - 3$ and for pairs of vertices on the same row it will be $r - 4$ or $r - 2$.

Constructed this way, G has degree $\Delta(G) = 2r - 3 - 1 = 2r - 4$, has order $r(r-1) = \frac{\Delta+4}{2}\frac{\Delta+2}{2} = \left(\frac{\Delta+3}{2}\right)^2 - \frac{1}{4}$ and each pair of vertices in G has at least one common neighbor.

When we start with $K_{r-2}\square K_r$, where $\Delta(K_{r-2}\square K_r) = 2r - 4$, we still have r columns and therefore $\frac{r(r-1)}{2}$ pairs of columns. By lemma 2.2 there is an arrangement of the pairs $\{12, \dots, 1r, 23, \dots, 2r, \dots, (r-1)r\}$ in $r - 1$ rows with each row being a permutation of $\{1,\ 2,\ \dots,\ r\}$. We take $r - 2$ of those rows and use such an arrangement to select the "horizontal" edges to be removed from $K_{r-2}\square K_r$ to obtain then a new graph H with the same vertex set, that is $V(H) = V(K_{r-2}\square K_r)$. After this procedure of edge removal, the amount of common neighbors will remain $r - 4$ for pairs of vertices on the same column and will be $r - 4$ or $r - 2$ for pairs of vertices on the same row. Constructed this way, H has degree $\Delta(H) = 2r - 4 - 1 = 2r - 5$, has order $r(r-2) = \frac{\Delta+5}{2}\frac{\Delta+1}{2} = \left(\frac{\Delta+3}{2}\right)^2 - 1$ and each pair of vertices in H has at least one common neighbor.

□

3 Attacking the problem from above

In this paragraph we establish some notation and preliminary facts to be used throughout the section. Consider a graph G such that each pair of its vertices has a common neighbor. Take a vertex $v_0 \in G$. We label its neighbors this way: $N(v_0) = \{v_1, v_2, \dots, v_d\}$, $d \leq \Delta$. Those vertices, having one subscript, will be referred to as vertices at the first level. As those are all the neighbors for v_0 and $N(v_0) \cap N(v_i)$ must be nonempty for every $i = 1, \dots, d$, we have that, given $1 \leq i \leq d$, necessarily $N(v_i) \supset \{v_0, v_{i'}\}$ for some $i' \neq i$. We will label the neighbors of each v_i this way: $N(v_i) = \{v_0, v_{i'}, v_{i1}, \dots, v_{id_i}\}$, where $d_i \leq \Delta - 2$. In such a graph G each pair of its vertices has a path of length 2 joining them. Therefore the vertices labeled above are already all the vertices of G. Thus

$$|V(G)| \leq 1 + d + d(\Delta - 2) \leq 1 + \Delta + \Delta(\Delta - 2) = \Delta(\Delta - 1) + 1.$$

We remark that if d is odd then, by the pigeonhole principle, there is a v_i with two neighbors in $N(v_0)$. Take such a vertex, labeled v_1. We will label its neighbors this way: $N(v_1) = \{v_0, v_2, v_3, v_{i1}, \dots, v_{id_1}\}$, where $d_1 \leq \Delta - 3$ and therefore the maximum number of vertices in G will be strictly lower than in the above calculation: for d odd, $|V(G)| \leq \Delta(\Delta - 1)$.

By the result of Hahn et. al (2002), for $\Delta \geq 3$, we have actually $|V(G)| \leq \Delta(\Delta - 1)$.

The lemma 3.1 below establishes that if a graph G, where any pair of its vertices has a common neighbor, with order $|V(G)| > (\Delta - 1)^2 + 1$ exists, then it must be regular.

Lemma 3.1. *Consider a graph G such that each pair of its vertices has a common neighbor. If G has order $|V(G)| > (\Delta - 1)^2 + 1$ then G is regular. If Δ is even then, for $|V(G)| > (\Delta - 1)^2$, G must be regular.*

Proof. Suppose G is not $\Delta-$regular. There exists a vertex v_0 with $deg(v_0) = d \leq \Delta - 1$. Thus,

$$|V(G)| \leq 1 + d + d(\Delta - 2) \leq 1 + \Delta - 1 + (\Delta - 1)(\Delta - 2) = 1 + (\Delta - 1)^2.$$

To achieve the maximum number of vertices, we'll have $N(v_0) = \{v_1, v_2, \ldots, v_{\Delta-1}\}$. If Δ is even, $\Delta - 1$ is odd and, in this case there is a vertex v_1 with $N(v_1) = \{v_0, v_2, v_3, v_{i1}, \ldots, v_{id_1}\}$, where $d_1 \leq \Delta - 3$. Therefore the maximum number of vertices in G will be strictly lower than in the above calculation: if G is not $\Delta-$regular and Δ even, $|V(G)| \leq (\Delta - 1)^2$. □

Next we prove that there exists no such graphs of order $|V(G)| = \Delta(\Delta - 1)$.

Theorem 3.2. *Consider a graph G such that each pair of its vertices has a common neighbor. If $\Delta(G) = \Delta$ odd, then $|V(G)| < \Delta(\Delta - 1)$.*

Proof. Suppose G has order $|V(G)| = \Delta(\Delta - 1)$. By the previous lemma G must be regular. Take a vertex v_0. $N(v_0) = \{v_1, v_2, \ldots, v_k\}$.

Consider Δ odd. If there was any pair of vertices on the first level, v_i, v_{i^*}, with a common neighbor on the second level then we would have less than $\Delta(\Delta-2)-1$ distinct vertices on the second level and therefore $V(G) \leq \Delta(\Delta - 1) - 1$. So we have that for any $2 \leq i \leq \Delta$, $N(v_i) = \{v_0, v_{i'}, v_{i1}, \ldots, v_{i(\Delta-2)}\}$ and $N(v_1) = \{v_0, v_2, v_3, v_{11}, \ldots, v_{1(\Delta-3)}\}$. The vertices having two subscripts are not at the first level and will be referred to as vertices at the second level. They are all distinct. As $\Delta - 2$ is odd, we will label v_{21} the vertex in $N(v_2)$ which has $N(v_{21}) \supset \{v_2, v_{22}, v_{23}\}$. For any other v_{2j} we have $N(v_{2j}) \supset \{v_2, v_{2j'}\}$. That means that the vertices $\{v_{21}, \ldots, v_{2(\Delta-3)}\}$ have together at most more $\Delta - 3 + (\Delta - 3)(\Delta - 2) = (\Delta - 2)^2 - 1$ neighbors. We must have $N(v_2) \cap N(v_{ij}) \neq \emptyset$ for every $3 \leq i \leq \Delta$ and $1 \leq j \leq \Delta - 2$ but that would mean that the vertices $\{v_{21}, \ldots, v_{2(\Delta-3)}\}$ have together at least more $(\Delta - 2)^2$ neighbors, a contradiction. Therefore, for odd Δ there is not such graph with $|V(G)| = \Delta(\Delta - 1)$.

□

It is a well-known result that for a Δ-regular graph G the product $\Delta|V(G)|$ must be even. Using this fact, with Theorem 3.2 and Lemma 3.1 we have the following corollary:

Corollary 3.3. *Consider a graph G such that each pair of its vertices has a common neighbor. If $\Delta(G) = \Delta$ is odd then $|V(G)| \leq \Delta(\Delta - 1) - 2$.*

4 A mutual acquaintance party problem

We state here a special case of the mutual acquaintance party problem under consideration:

***Mutual Acquaintance Party Problem** ($\Delta = 5$): "How many people can we have in a party where any two people have a mutual acquaintance and each person has at most 5 acquaintances? "*

A graph G is a valid configuration for such a party if and only if $\Delta(G) \leq 5$ and each pair of vertices in G has a common neighbor. For $\Delta = 5$ the general upper bound obtained in Corollary 3.3 yields $|V(G)| \leq 18$.

If such a graph with 18 vertices exists, it must be regular by Lemma 3.1, because $(\Delta - 1)^2 + 1 = (5 - 1)^2 + 1 = 17 < 18$.

Now, for $\Delta(G) = 5$ if such a graph with $|V(G)| = 17$ existed it couldn't be regular. In the next result, lemma 4.1, we prove that such graph does not exist.

Lemma 4.1. *There is not a graph of diameter two with every edge on a triangle with $\Delta(G) = 5$ and $|V(G)| = 17$.*

Proof. As G cannot be regular and $\Delta(G) = 5$, there is a vertex v_0 with less than $\Delta = 5$ neighbors. As $|V(G)| = 17 = (\Delta - 1)^2 + 1$, there is only one such vertex v_0 and it must have $\Delta - 1 = 4$ neighbors (if not G would have less than $(\Delta - 1)^2 + 1$ vertices). We denote $N(v_0) = \{v_1, v_2, v_3, v_4\}$. For $1 \leq i \leq 4$ we denote $N(v_i) = \{v_0, v_{i'}, v_{i1}, v_{i2}, v_{i3}\}$. Without loss of generality, consider $v_1 \in N(v_2)$ and $v_3 \in N(v_4)$. As $\Delta = 5$ is odd, we label v_{11} the vertex of $N(v_1)$ which has two neighbors in $N(v_1)$. We denote $N(v_{11}) \supset \{v_1, v_{12}, v_{13}\}$. As $N(v_{11}) \cap N(v_i)$ must be nonempty for $3 \leq i \leq 4$, necessarily $N(v_{11}) = \{v_1, v_{12}, v_{13}, v_{3j_3}, v_{4j_4}\}$. We remark that v_{3j_3} and v_{4j_4} must be neighbors. Analogously for $N(v_{21})$: $N(v_{21}) = \{v_2, v_{22}, v_{23}, v_{3\ell_3}, v_{4\ell_4}\}$. The same reasoning applies to $N(v_{31})$ and $N(v_{41})$.

As $N(v_{11}) \cap N(v_{21})$ must be nonempty, necessarily $v_{3j_3} = v_{3\ell_3}$ or $v_{4j_4} = v_{4\ell_4}$. Therefore $N(v_{3j_3}) = \{v_3, v_{31}, v_{11}, v_{12}, v_{4j_4}\}$ and $N(v_{4j_4}) = \{v_4, v_{41}, v_{11}, v_{12}, v_{3j_3}\}$.

If it was true that $N(v_{11}) \cap N(v_{21}) = \{v_{3j_3}\}$ we would have $N(v_{3j_3}) \supset \{v_3, v_{31}, v_{11}, v_{12}\}$, and to $N(v_{11}) \cap N(v_{3j_3})$ and $N(v_{21}) \cap N(v_{3j_3})$ to be both nonempty v_{3j_3} should have two more distinct neighbors, a contradiction.

The only other possibility is $N(v_{11}) \cap N(v_{21}) = \{v_{3j_3}, v_{4j_4}\}$. We have $N(v_{3j_3}) = \{v_3, v_{31}, v_{11}, v_{12}, v_{4j_4}\}$ and $N(v_{4j_4}) = \{v_4, v_{41}, v_{11}, v_{12}, v_{3j_3}\}$. Now consider $v_{4j_2} \notin N(v_{4j_4})$. Then for $N(v_{4j_2}) \cap N(v_{3j_3})$ to be nonempty, v_{3j_3} must have another neighbor, a contradiction. □

For the special case at hand, $\Delta = 5$, we can modify the method from the previous section for constructing large graphs where every pair of vertices has a common neighbor starting from a cartesian product of complete graphs: starting with $K_4 \square K_4$, which has $\Delta(K_4 \square K_4) = 6$, we have removed edges in this way: one horizontal edge per row and one vertical edge per column, removing a total of 8 edges which enabled us to reduce by 1 the degree of all 16 vertices. As a result we constructed the following graph that has $\Delta = 5$, order $|V(G)| = (\Delta - 1)^2 = (5 - 1)^2 = 16 > \frac{\Delta^2 + \Delta}{2}$, and each pair of its vertices has a common neighbor.

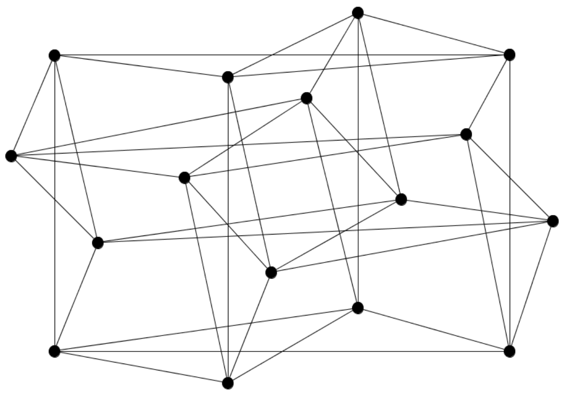

FIGURE 1. This graph is obtained by removing edges from $K_4 \square K_4$, one horizontal edge per row and one vertical edge per column.

By the results obtained here and by checking the smaller cases ($\Delta \leq 4$) by hand, we have the following table:

Δ(G)	$\mathbf{\lvert V(G)\rvert_{max}}$	**Graph realizing**
2	3	$C_3 = K_3$
3	4	$K_4 - e$ (a.k.a. diamond)
4	9	$K_3 \square K_3$ (3×3 lattice graph)
5	16 or 18	×
$4b - 1 \geq 7$	between $\left(\frac{\Delta+3}{2}\right)^2 - 1$ and $\Delta(\Delta - 1) - 2$	×
$4b \geq 8$	between $\left(\frac{\Delta+3}{2}\right)^2 - \frac{1}{4}$ and $\Delta(\Delta - 1)$	×

Therefore, for the case $\Delta = 5$ of the mutual acquaintance problem considered here, we have narrowed the solution to two possibilities as we have constructed a configuration with 16 vertices but still have not constructed a configuration with 18 vertices nor ruled out this possibility.

For the next case, $\Delta = 7$, we already have a significant gap: the solution is between 24 and 40.

5 Conclusion

Graphs of diameter two with every edge in a triangle contain as a subclass the highly structured strongly regular graphs with nonzero parameters. It is still unknown if there exists a strongly regular graph with nonzero parameters with the largest possible order, $n = (k-1)^2$. We believe that results in this class, that has more flexibility in its definition, can contribute to solving this longstanding problem.

They are also obviously contained in the class of graphs of diameter two. We believe that some problems and results about diameter two graphs should be simpler or sharper if restricted to graphs where every edge is in a triangle.

We stress that obtaining general upper bounds on the degree-diameter problem, even for diameter two, even when restricted to some subclasses, does not seem to be an easy task (see the survey by Miller and Siran, (2005)). Attacking the degree-diameter problem from below is much more usual. Nevertheless the construction of infinite families is important for studying asymptotics on the degree-diameter problem.

Graphs where any pair of its vertices has a common neighbor are exactly those with adjacency matrices whose square is positive. The results here have a straightforward interpretation in terms of nonnegative, symmetric, zero-diagonal matrices whose square is positive. This is part of a work in progress.

We hope to draw attention to this subject and to stimulate other researchers to test their techniques for diameter 2 graphs, strongly regular graphs, nonnegative matrices and any other related areas in this class.

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