

A Discrete Monitoring Method for Pricing and Hedging Asian Interest Rate Options - the Brazilian IDI option case

Allan Jonathan da Silva

Laboratório Nacional de Computação Científica
Av. Getúlio Vargas, 333, Quitandinha, Petrópolis - CEP: 25651-075
allanjs@lncc.br

Jack Baczynski

Laboratório Nacional de Computação Científica
Av. Getúlio Vargas, 333, Quitandinha, Petrópolis - CEP: 25651-075
jack@lncc.br

José V. M. Vicente

Banco Central do Brasil
Avenida Presidente Vargas, 730 - Centro, Rio de Janeiro - RJ, 20071-000
jose.valentim@bcb.gov.br

ABSTRACT

We propose a new numerical finite difference method to replace the classical finite difference schemes of financial engineering used to solving PDEs (see e.g. Tavella (2002) and Duffy (2004)). The motivation for doing so stems from the fact that spurious oscillation solutions occur when volatilities are low (i.e., when the Peclet number is high) and serious collateral matters appear in the attempts to correct the problem. Actually, low volatilities are the range that we observe in the interest rate markets and, unlike the classical schemes, our method covers the whole spectrum of volatilities in the interest rate dynamics. The method we developed is reliable and highly competitive. It straightforwardly adapts to other interest rate derivative securities than the IDI (Interbank Deposit Rate Index) option, e.g., bond options, swaptions and caps. Via minor changes the method fits other types of path-dependent options, as well as Coupon bonds, bond options and callable bonds.

KEYWORDS. Interest Rates Derivatives, IDI Option, PDE Option Pricing.

Main Area: GF - Financial Management, MP - Probabilistic Models, SIM - Simulation

1. Introduction

We propose a new numerical finite difference method to replace the classical finite difference schemes of financial engineering used to solving PDEs (see e.g. Tavella (2002), Duffy (2004) and Burden & Faires (2010)). The motivation for doing so stems from the fact that spurious oscillation solutions occur when volatilities are low (i.e., when the Peclet number is high) and serious collateral matters appear in the attempts to correct the problem. Actually, low volatilities are the range that we observe in the interest rate markets and, unlike the classical schemes, our method covers the whole spectrum of volatilities in the interest rate dynamics.

Our method is devised as a version of the full implicit method (see, e.g., Tavella (2002), Wilmott (2006)), and extended to provide hedges along with prices. One of the modifications we introduced is inspired in a technique that appears in Milev & Tagliani (2013). It is worth reminding that the method of Milev & Tagliani (2013) adapts to the Black-Scholes dynamics, while ours fits the interest rate derivatives with Vasicek (1977), CIR (Cox et al. (1986)) and other types of short-rate models. Our numerical scheme is second order accurate in space, consistent, and stable under mild conditions. We shall name it Modified Full Implicit (Interest Rate) Method.

We attest the good performance of the method, pricing a zero-coupon bond and another type of interest rate derivative security named IDI (Interbank Deposit Rate Index) option, both in the Vasicek dynamic. Namely, a convergence analysis was done, as we consider both continuously compounding and daily compounding rate of interest to model the money market account and the updating of the IDI path.

Remark The IDI option is a financial option of Asian type and, as such, the payoff depends on the path followed by the short term interest rate. It presents cheaper prices than the standard options and it is less sensitive to extreme market conditions that may prevail close to the expiration day - due to random crashes or outright manipulation. So, it is commonly used by corporations to manage interest rate risk. Actually, it is a standardized derivative product traded at the Securities and Futures Exchange in the Brazilian fixed income market.

The ID index updating is built up discretely based on the overnight DI rate, which is an annualized rate over one day period. It is calculated and published daily, and represents the average rate of the inter-bank overnight transactions (Brace (2008)). Based on a Martingale approach, closed form solutions to price an IDI contract are available in the literature, assuming for mathematical tractability reasons that the updating of the IDI is continuous in time. In this scenario, one-factor models were developed in Vieira & Pereira (2000) and Barbachan & Ornelas (2003) to price the IDI Option via the short rate dynamics as given in Vasicek (1977) and Cox et al. (1986), respectively. A Multi-factor Gaussian model was developed in Almeida & Vicente (2012) to price the IDI Option and Bond prices. Also, Genaro & Avellaneda (2013) proposed to incorporate the potential changes in the targeting rates via pure jump process.

Carrying on the evaluation of our finite difference scheme, we acknowledge its advantages considering the following approaches on a pricing problem of an IDI call option with the Vasicek dynamic.

- we obtain the estimates of the prices (and hedges) according to the Modified Full Implicit method, and consider updating the IDI path discretely. This updating rule allow us to track realistically the evolution of the index and to achieve the exact pay-off representation.
- we obtain the prices via the closed form expressions given in Vieira & Pereira (2000), assuming a continuously compounding rate of interest, which is actually an idealization for mathematical tractability.

So, our approach corresponds to obtaining approximate prices for the exact problem (with respect to the payoff) while that of Vieira & Pereira (2000) corresponds to obtaining an exact price for the

approximate problem. The results this analysis provides are interesting and corroborate the Tankov and Cont Tankov & Cont (2003) conjecture, which asserts that, typically, the former scenario yields better results than the latter.

Indeed, via numerical simulations, we observed meaningful relative discrepancies in the prices for some prescribed examples whose parameters are good representatives of the market. So, using one or other method plays a difference. Now, neither prices represent a benchmark - which should correspond to a framework that models the IDI discretely and provides the exact solution for the price. However, the Modified Full Implicit Method can be refined approaching the benchmark. On the other hand, any short rate model which adopts the IDI continuously compounded hypothesis (as in Vieira & Pereira (2000), Barbachan & Ornelas (2003), Almeida & Vicente (2012) and Genaro & Avellaneda (2013)) are obviously inconsistent with refinements with respect to the index updating, so they cannot approach the benchmark.

Reminding that the continuously updating procedure for calls boils down to a more expensive payoff than the discretely updating one, it is reasonable to expect prices to be more expensive in the former than the latter procedure. The simulations have ascertained more that this in fact. They showed that, starting with a reasonable refined mesh, our call prices are cheaper than those of the continuously updating case of Vieira & Pereira (2000) and, as the mesh is refined, our prices move further downwards approaching the benchmark - and away from the prices of Vieira & Pereira (2000). Analogous conclusions are also obtained with put options.

The study carried out in this paper, in conjunction with the numerical simulations performed with the above derivatives, indicate, in fact, that the method we developed is reliable and highly competitive - not to say exceeding among other interest rate methods. It straightforwardly adapts to other interest rate derivative securities, e.g., bond options, swaptions, caps and floors, adjusting the appropriate terminal condition. Via minor changes at the functions assigned to the jump conditions, the method fits other types of path-dependent options, as well as Coupon bonds, Coupon bond option and callable bonds. The study immediately suggests that the realistic discretely compounding scheme (associated with the Modified Full Implicit Method to obtain price estimates) give best results in the interest rate scenario, in detriment of the continuously compounding scheme.

We organize the article as follows: In Section 2 we present the motivation of the discretely daily monitoring approach. In section 3 we present the partial differential equation that will be used in pricing the IDI call option and justify a coordinate transformation for the PDE. In Section 4 we propose a scheme that is consistent and stable to convective dominant parabolic equations. Section 5 presents the results and Section 6 concludes the article.

2. Problem statement (versus the problem of Vieira & Pereira (2000))

We consider the problem of pricing and IDI option, assuming that the ID index y accumulates discretely according to

$$y(t_n) = y(t_0) \prod_{i=1}^n (1 + DI(t_{i-1}))^{\frac{1}{252}}, \quad n = 1, \dots, N, \quad (1)$$

where t_i denotes the end of day i and $DI(\cdot)$ assigns the DI rate, i.e., the average of the interbank rate of a one-day-period, calculated daily and expressed as the effective rate per annum. A detailed definition of the DI rate can be found in Brace (2008). This procedure is consistent with the BM&FBovespa protocols. Correspondingly, the discretely monitored pay-off for the call option with maturity in $T = t_N$ is given by

$$\max(y(t_N) - K, 0). \quad (2)$$

We also suppose that the instantaneous short-term interest rate r —which shapes the DI rate, evolves according to Vasicek model (see Vasicek (1977))

$$dr(t) = a(b - r(t))dt + \sigma dW_t. \quad (3)$$

This Ornstein-Uhlenbeck stochastic process pulls the short rate to a level b at a rate a against with a normally distributed random term σdW , where dW is a standard Brownian Motion.

For the sake of comparison, the usual setup found in the literature assumes that the IDI index accumulates continuously according to

$$y(T) = y(t)e^{\int_t^T r_u du}, \quad (4)$$

instead of (1). Correspondingly, the continuously monitored pay-off for the call option with maturity in T is given by

$$\max(y(T) - K, 0), \quad (5)$$

which stands as the counterpart of (2). Hence, concerning this important aspect, our framework is more realistic than that usually found in the literature. Under this continuously compounded hypothesis, Vieira & Pereira (2000) developed a closed-form solution for the price of an ID call option with maturity in T , where the short rate also follows the Vasicek model.

Under the above hypothesis, zero-coupon bond prices are very similar to the daily compounded interest rates. However this is not the case when dealing with assets as Asian interest rate options. Hence, a main question in this paper is how good or bad are both models in pricing theoretical IDI Options.

For later use, the price of an IDI call option with maturity in T at time t , in the continuously compounded hypothesis, is given by Vieira & Pereira (2000). The price at time t of a zero-coupon bond that pays 1 at time T is presented by Vasicek (1977).

3. PDE formulation

Our aim is pricing a contract u that is a function of three variables, say the time t , the random shocks of the interest rates r and the ID index that accumulates daily payments y . We assume that $u(t, r(t), y(t)) \in C^{1,2,0}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$. Following the steps of Bollen (1997) and applying Ito's lemma (see e.g. Oksendal (2007)), we set up a portfolio π containing two similar contracts with different maturities, obtaining

$$d\pi_t = \frac{\partial u_1}{\partial t} dt + \frac{\partial u_1}{\partial r} dr + \frac{\sigma^2}{2} \frac{\partial^2 u_1}{\partial r^2} dt - \Delta \left(\frac{\partial u_2}{\partial t} dt + \frac{\partial u_2}{\partial r} dr(t) + \frac{\sigma^2}{2} \frac{\partial^2 u_2}{\partial r^2} dt \right) \quad (6)$$

In spite of the fact that we are modeling a path-dependent option, the portfolio (6) exhibits a classical shape. This is so because the stochastic differential equation for the IDI degenerates, in the sense that $dy = 0$. We remind that the quantity given by (1) changes only at a set of discrete jump times $\Omega = (t_1, \dots, t_N)$ that represents the end of the trading days.

Let the market price of risk be $\lambda = 0$. The usual no-arbitrage argument implies that the price of the IDI option $u = u(t, r, y)$ at time $t \notin \Omega$, i.e., when the IDI remains constant, is given by

$$\frac{\partial u}{\partial t} + a(b - r(t)) \frac{\partial u}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial r^2} = r(t)u. \quad (7)$$

Across each $t_n \in \Omega$, absence of arbitrage ensure that the price of the option is continuous (Wilmott (2006), Zvan et al. (1999)). This is mathematically represented by the following jump condition:

$$u(t_n - \epsilon, r, y^-) = u(t_n + \epsilon, r, y^+), \quad (8)$$

where $y^+ = y^- [(1 + r)^{\frac{1}{252}}]$, and $0 < \epsilon \ll 1$. We could alternatively derive the PDE (7) by using the Discounted Feynman-Kac Theorem (Oksendal (2007)).

To ensure uniqueness of solution we prescribed arbitrary functions to describe how the PDE must behave at the extremes of the domain. In the case of the IDI Option we chose homogeneous Neumann boundary condition. We know that the dynamics (3) allows negative and positive

infinite values for r with non-zero probabilities. Hence, this conditions ensures that an infinitesimal change in r at the boundaries does not change the value of the option. This is intuitive because the IDI Option price is actually insensitive to changes in extreme negative or positive values of r . This fact can also be verified in the closed-form formula of Vieira & Pereira (2000). The terminal condition is the pay-off of the call option is

$$u(T, r, y) = \max(y - K, 0), \quad (9)$$

where K is the strike price and y is viewed as (2).

As it occurs with the Stock-Asian-Parisian Options (Zvan et al. (1999)), we have that away from monitored times the PDE (7) has no y dependence. The terminal condition (9) implies that a set of independent one-dimensional PDEs must be solved. The IDI Option price is calculated via (7) backwards in time from the terminal condition (9) up to the first $t_n \in \Omega$. We apply, in the sequel, the jump condition to find the option value at t_n^- . Using these values as the new terminal condition we repeat the process $N + 1$ times to meet the current value of the option, where N is the cardinality of the set (Ω).

3.1. Coordinate transforms

Finding a solution to (7) is a well-known problem in physics and in finance. Numerically speaking, it is inconvenient if the sign of the convective term changes and the volatility is very low, which are common facts in interest rate derivatives. Financially speaking, it is undesirable to have the same precision for all points in the grid, because we are pricing a product based on the current interest rate. So, in the same lines as in Tavella (2002), we first propose a change of variable that allows us to retrieve the solution in a nonuniform grid in r , which becomes thinner in some desirable or needed region. Then we appropriately modify a finite difference scheme to overcome the drawbacks of a convective dominant PDE.

The "proximity" of the nonzero probability left boundary to the actual level of interest rates suggests that small errors at the left boundary lead to inaccurate results near the strike price, where a sharp gradient occurs in conditional derivatives. So we specified a new variable $x = \ln(rd + c)$, where c and $d > 0$ are constants such that $c > |\min(r)|d$.

So, we get the transformed PDE in the new coordinate x

$$\frac{\partial u}{\partial t} + \left[\frac{a(db - e^x + c)}{e^x} - \frac{\sigma^2 d^2}{2e^{2x}} \right] \frac{\partial u}{\partial x} + \left(\frac{\sigma^2 d^2}{2e^{2x}} \right) \frac{\partial^2 u}{\partial x^2} = \left(\frac{e^x - c}{d} \right) u. \quad (10)$$

The terminal condition does not depend on r directly and remains the same. The new jump condition is obtained substituting the value of r in the new coordinate.

4. Modified Full-Implicit Method

We now address the above problems via finite difference methods. This class of methods consists in the discretization of the x domain over some finite interval $[x_{min}, x_{max}]$ with J points and in approximating the derivatives of the PDE by its incremental ratio Δx , which converges to the derivative as $\Delta x \rightarrow 0$. The method consists in replacing the derivatives in (11) by its numerical values in a finite number of points (Tavella (2002) and Iserles (2009)). To overcome the stability restrictions of the classical finite difference methods, we introduce an appropriately chosen function $f = f(\sigma, b, a, c, x, \Delta r)$ in the Full-Implicit scheme, given by

$$f = \frac{1}{2\delta} \left(\frac{a(b + c + 1)}{e^x} + \frac{d^2 \sigma^2}{2e^{2x}} + 1 \right), \quad (11)$$

which, in conjunction with a new reaction term prescribed as

$$G_j = \left(\frac{e^{x_j} - c}{d} \right) u_j^{n+1} + 4f_j u_j^{n+1} - 2f_j (u_{j-1}^{n+1} + u_{j+1}^{n+1}), \quad (12)$$

yields the modified version of the full-implicit discretization, namely

$$\partial_t^- u_j^{n+1} + \mu_j \partial_x^0 u_j^{n+1} + S_j \partial_x^+ \partial_x^- u_j^{n+1} = G_j \quad (13)$$

and the corresponding system of equations

$$u^{n+1} = \frac{1}{\Delta t} \bar{P}^{-1} u^n. \quad (14)$$

In this case $\bar{P} = (\bar{p}_{ij})$ turns out to be such that

$$\bar{p}_{j,j-1} \leq 0, \quad (15)$$

$$\bar{p}_{j,j} > 0 \quad \text{and} \quad (16)$$

$$\bar{p}_{j,j+1} \leq 0. \quad (17)$$

Moreover,

$$\bar{p}_{j,j} > \sum_{j \neq i} |\bar{p}_{i,j}|, \quad (18)$$

so that \bar{P} becomes an M-matrix.

A similar idea has been suggested in Milev & Tagliani (2013) for the stock-options case. The following proposition shows that the modified version of the Full-Implicit scheme is in fact spurious oscillation free.

Theorem 1 *The matrix \bar{P} is in compliance with inequalities (16), (17) and (18) and is strictly diagonally dominant.*

Proof Relying on the modified version of the Full-Implicit method associated with PDE (11), and bearing in mind that μ and S are given by

$$\mu = \left(\frac{a(db - e^x + c)}{e^x} - \frac{\sigma^2 d^2}{2e^{2x}} \right) \quad (19)$$

and

$$S = \left(\frac{\sigma^2 d^2}{2e^{2x}} \right), \quad (20)$$

it follows that

$$0 \geq \left[\frac{\mu_j}{2\Delta x} - \frac{S_j}{\Delta x^2} - 2f_j \right], \quad (21)$$

$$0 < \left[\frac{1}{\Delta t} + \frac{2S_j}{\Delta x^2} + \frac{e^{x_j} - c}{d} + 4f_j \right] \quad \text{and} \quad (22)$$

$$0 \geq \left[-\frac{\mu_j}{2\Delta x} - \frac{S_j}{\Delta x^2} - 2f_j \right]; \quad (23)$$

for any choice of $a, b, \sigma \geq 0$, $-\frac{c}{d} < r$ and some $\delta \ll 1$.

From the definition of M-matrices (Plemmons (1977)), \bar{P} is an M-matrix, so that $\bar{P}^{-1} \geq 0$. So, the solution u provided by the finite difference scheme (14) is positivity-preserving: negative prices are precluded.

4.1. Stability

The next propositions show that u is stable and a non-increasing function in $t \in [0, T]$. Firstly, let us state an auxiliary lemma.

Lemma 1 Assume that Q is diagonally dominant by rows and set $\alpha = \min_k (|q_{kk}| - \sum_{j \neq k} |q_{kj}|)$. Then $\|Q^{-1}\|_\infty < \frac{1}{\alpha}$.

Proof See Varah (1975).

Proposition 1 Under the (very) mild condition

$$\frac{1}{\Delta t} \frac{1}{\left| \frac{1}{\Delta t} + \frac{2S}{\Delta x^2} + \frac{e^x - c}{d} + 4f - \frac{\mu}{\Delta x} \right|} < 1 \quad (24)$$

the solution u is stable. So, it is spurious oscillations free and we say that u is conditional stable.

Proof The left side of (25) is an upper bound for the spectral radius of the iteration matrix $(\frac{1}{\Delta t} \bar{P}^{-1})$.

Proposition 2 The solution of (15) satisfies the maximum principle.

Proof Applying the sup-norm $\|\cdot\|_\infty$, using Lemma 1 and the conditional stability property of u , we have

$$\begin{aligned} \|u^{n+1}\|_\infty &= \frac{1}{\Delta t} \|\bar{P}^{-1}u^n\|_\infty \\ &\leq \frac{1}{\Delta t} \frac{1}{\left| \frac{1}{\Delta t} + \frac{2S}{\Delta x^2} + \frac{e^x - c}{d} + 4f - \frac{\mu}{\Delta x} \right|} \|u^n\|_\infty \\ &\leq \|u^n\|_\infty \end{aligned}$$

4.2. Consistency and Convergence

Theorem 2 The modified full-implicit method associated with PDE (14) is of order of accuracy $\mathcal{O}(\Delta t, \Delta x^2)$.

Proof The Taylor series for (13) boils down to

$$\frac{(e^{x_j} - c)}{d} u(t_n, x_j) + \frac{(e^{x_j} - c)}{d} \Delta t \frac{\partial u}{\partial t}(\xi_t, x) - f(\Delta x)^2 \left(\frac{\partial^2 u}{\partial x^2}(t, \xi'_x) + \frac{\partial^2 u}{\partial x^2}(t, \xi''_x) \right),$$

where $t_n \leq \xi_t \leq t_{n+\Delta t}$, $x_j \leq \xi'_x \leq x_{j+\Delta x}$ and $x_{j-\Delta x} \leq \xi''_x \leq x_j$. It follows that (13) has order of accuracy $\mathcal{O}(\Delta t, \Delta x^2)$. By its turn, relying on the definitions of $\partial_t^- u_j^{n+1}$, $\partial_x^0 u_j^{n+1}$ and $\partial_x^- \partial_x^+ u_j^{n+1}$, it follows that the left hand side of (14) also has order of accuracy $\mathcal{O}(\Delta t, \Delta x^2)$.

The Lax Theorem states that if a finite difference scheme is consistent (e.g., in the sense of proposition 2) and stable, then it is convergent (Tavella (2002)).

5. Numerical results

In this section, we shall address two pricing problems in the Vasicek dynamic. The former aims to attest the good performance and the properties of the Modified Full-Implicit method as described in Section 4, addressing a zero-coupon bond and the IDI option. The latter aims a comparative analysis addressing the prices of the IDI option according to our approach and that of Vieira & Pereira (2000).

5.1. Convergence study

Assuming the continuously compounding rate of interest, we calculate the discrepancy between the price $P(r_j, 0, T)$ of the bond given by the closed-form expression of Vasicek (1977) and the price u_j^0 given by the Modified Full-implicit method with the prescribed terminal condition of a zero-coupon bond. The error measure we adopt is

$$\epsilon = \sqrt{\frac{1}{N} \sum_{j=1}^N (u_j^0 - P(r_j, 0, T))^2}, \quad (25)$$

where the subscript j assigns the spatial grid of the interest rates.

In the simulations we use $\Delta t = 0,00099206$ (four time-steps per day). Table 1 illustrates discrepancies for solutions with 200, 400, 600 and 800 spatial grid points and parameters set as $a = 0.1$, $b = 0.1$ and $\sigma = 0.02$ in a 1-year zero-coupon bond price problem. The columns 2 and 3 show that if $\epsilon \rightarrow 0$ then $\Delta x \rightarrow 0$.

To numerically estimate the order of convergence of the method, let us find q such that

$$\epsilon \leq C \Delta x^q, \quad (26)$$

for constant C . The table 1 shows that $q = 2.03$ in the domain of interest $r \in (-0.1, 0.5)$. We also tested the convergence rate of the method using the IDI option assuming a daily updating. Since, in this case, the limit value of the price is not available, we look at ratios of differences between u_j^0 computed for different J 's, given by

$$q = \log_2 \left[\frac{u_J^0 - u_{\frac{J}{2}}^0}{u_{\frac{J}{2}}^0 - u_{\frac{J}{4}}^0} \right]. \quad (27)$$

For $r_j = 0.1$ and starting with $J = 1600$ grid points we obtained $q = 2.02$. To confirm the performance of the method for the case where the limit value of the price is not available, we replicated the above procedure for the zero-coupon bond. Again, a consistent rate was obtained, namely $q = 1.997$. All the results above corroborate the early consistency analysis and the good performance of the method.

As a start-up to the next section, where our main results appear, we consider two zero-coupon bond pricing problems, where the sole difference between them is adopting, in one problem, a daily compounding interest rates and, in the other, a continuously compounding interest rates. Again the Modified Full-implicit method is applied, now in conjunction with the algorithm described at the end of section 3. In this particular example, the relative discrepancy (defined in the same lines as in equation (29) below) did not exceed 10% in the whole interest rate domain. In contrast to this, we shall see that the prices in the IDI case differ significantly if one adopts the continuously or the discretely updating.

The small discrepancies we found here are in fact a well known result when interest rates are deterministic. However, we believe this is the first time this result is observed for stochastic interest rates following the Vasicek dynamic.

We remind that, in the zero-coupon bond case, the discretely compounding yields can be straightforwardly obtained from the continuously compounding case. However, this technique can be helpful to price complex types of interest rate derivatives with discrete compounding, such as callable bonds.

5.2. Pricing

We compare the prices of IDI call options under the Vasicek model, considering the following approaches:

Table 1: Modified full-implicit method spatial convergence rate

N	Δx	ϵ	$-\log(\Delta x)$	$-\log(\epsilon)$
200	0.02482	0.01295	1.60509	1.88770
400	0.01244	0.00327	1.90504	2.48572
600	0.00830	0.00142	2.08078	2.84924
800	0.00623	0.00078	2.20553	3.10777

- we obtain the estimates of the prices according to the Modified Full Implicit method and the coordinate transformation afore mentioned, and consider updating the IDI path discretely. This updating rule allow us to track realistically the evolution of the index and to achieve the exact pay-off representation.
- we solve the closed form expressions given in Vieira & Pereira (2000) for the prices, assuming a continuously compounding rate of interest, which is actually an idealization for mathematical tractability.

The numerical results of cases I and II are summarized in figures 1 and 2, respectively, where we set $a = 0.1265$, $b = 0.0802$ and $\sigma = 0.0218$ in the Vasicek model. This calibration stemmed from the Brazilian overnight interest rate data from 2002 to 2014 and was produced via the General Method of Moments (Chan et al. (1992)). The initial value of the IDI is 100.000 points. The Modified Full Implicit method is used with 800 grid points for the ID index and a spatial mesh of 400 grid points for the interest rate. We use 5 steps per day with a daily jump condition at the last step, which satisfy the mild stability conditions required. Cases I and II refers to a call option. Case I (resp. II) have maturity in 252 days (resp. 504 days) and strike $K=109.550$ points (resp. $K=122.000$ points). Case III is summarized in figure 3, where we set $a = 0.2$, $b = 0.1$ and $\sigma = 0.1$ in the Vasicek model and a short maturity of 20 days. We set several refinements in this case for spatial mesh sizes with 50; 150; 250; 400 and 600 points.

First thing to notice from the numerical data is that, unlike the zero-coupon scenario previously mentioned, the relative discrepancy between prices obtained from the approaches under concern are not negligible at all, even with low volatilities (cases I - II) and short maturities (case III). The relative discrepancy between prices is here defined as

$$\theta_j = \frac{(\eta(r_j, 0, T) - u_j^0)}{u_j^0}, \quad (28)$$

where η stands for C or II . So, for $r_j = 10\%$, we have $\theta_j = 45.88\%$ and $\theta_j = 50.96\%$, for cases I and II. We recall that in the zero-coupon scenario, such relative difference did not exceed 10%. So, using one or other method plays a difference. Notice that neither prices represent a benchmark - which should correspond to a framework that models the IDI discretely and provides the exact solution for the price. However, the Modified Full Implicit method can be refined approaching the benchmark. On the other hand, any short rate modeling framework which adopts the IDI continuously compounded hypothesis - which is the case of Vieira & Pereira (2000), are obviously inconsistent with refinements with respect to the index updating, so they cannot approach the benchmark.

Reminding that the discretely updating procedure for calls boils down to a cheaper payoff than the continuously updating one, we expect prices to be cheaper in the former than the latter procedure, for a reasonable mesh refinement. Figures 1 to 3 show this indeed. In Figure 3 we may observe the downward movement of the prices as the spatial mesh sizes are refined in a sequence of 100, 150, 250, 400 and 600 points, leading the solutions towards cheaper call option prices, which actually represent the benchmark. So, even with reasonable refined meshes, our call prices are

cheaper than those of the continuously updating case of Vieira & Pereira (2000). As the mesh is refined, our prices move further downwards, approaching the benchmark and, simultaneously, moving further away from the prices of Vieira & Pereira (2000).

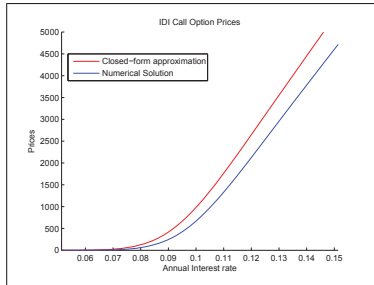


Figure 1: 1 year Call Option Prices

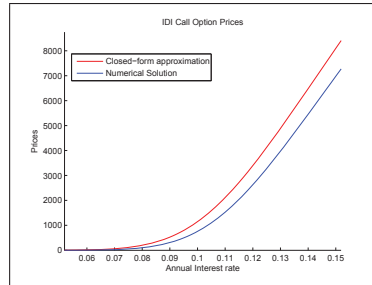


Figure 2: 2 year Call Option Prices

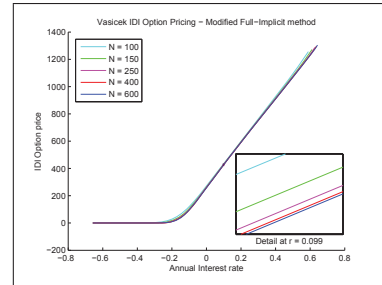


Figure 3: Prices with different meshes

5.3. Hedging results

In order to investigate the effect of the continuous updating hypothesis in the hedging process we now provide the delta of the IDI call option. By the fact that the IDI is an index and not a physical asset which we can buy and sell, we will evaluate the infinitesimal changes in the price of the option (given by Vieira & Pereira (2000)) with respect to a change in the whole term-structure. This is to use the asset $P(t, T)$ to replicate the IDI path.

It can be shown that deriving C with respect to $P(t, T)$, gives

$$\frac{\partial C}{\partial P(t, T)} = -K\Phi(h - k). \quad (29)$$

We also develop a version of (30) to be used numerically independent of a known closed-form formula for the price. Let us suppose the price of an IDI call option $C(y(t), P(t, T))$ depends on the actual level of the index $y(t)$ and the price of a bond with maturity equal to the option's maturity. So

$$\frac{\partial C}{\partial r} = \frac{\partial C}{\partial y(t)} \frac{\partial y(t)}{\partial r} + \frac{\partial C}{\partial P(t, T)} \frac{\partial P(t, T)}{\partial r}. \quad (30)$$

Using the zero-coupon bond $P(t, T)$ to hedge the IDI call option position, the Δ is calculated as

$$\Delta(t) = \frac{\frac{\partial C}{\partial r} - \frac{\partial C}{\partial y(t)} \frac{\partial y(t)}{\partial r}}{\frac{\partial P(t, T)}{\partial r}}. \quad (31)$$

The left panel of the figure 4 shows the portfolio values and the call option prices evolution through time adopting the continuous updating hypothesis. It is observed an almost perfectly replication strategy. The delta values of the right panel is calculated as in equation (30). The delta in the negative field means that we must to sell a quantity equal to $P(t, T)$ times $|\Delta|$ and deposit the proceeds in a bank account earning the risk-free rate.

If all these operations were performed continuously the difference between $P(t, T)$ times Δ and the issuer's bank account will be equal the difference between the index and the strike at maturity and the hedging error is zero. Henceforward we call hedging error the final cost of the hedging process, that is, the difference between the portfolio and the option price.

In figure 5 is shown the hedging errors with 10000 simulation paths for the two IDI updating methodologies using the above Vasicek delta closed-form formula: the former panel shows the results regarding continuously compounded index which results in a zero mean relative hedging

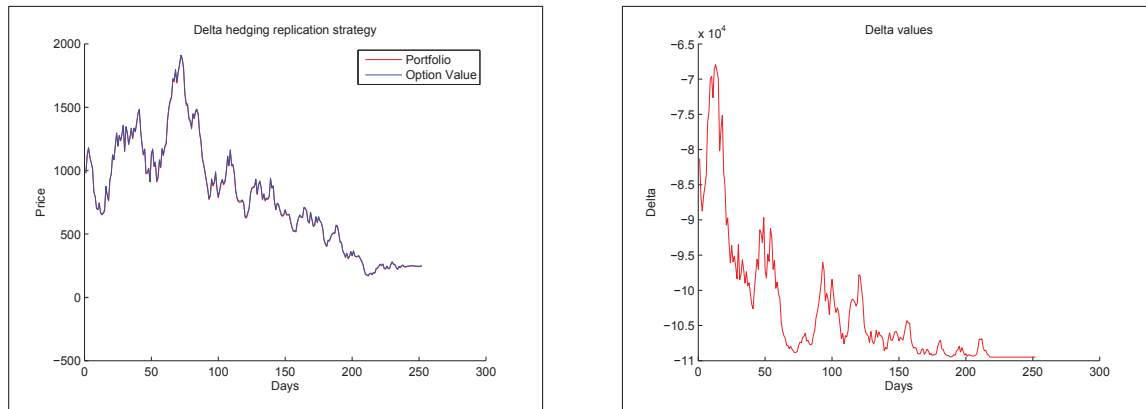


Figure 4: *Replicating strategy and delta values*

$IDI(0) = 100000$, $Strike = 109500$, $a = 0.1265$ $b = 0.0802$, $\sigma = 0.0218$, $T = 252$ days

error, the middle panel shows a relative hedging error that is in accordance with the experiments of the last section, say 41%, while the latter shows the analytical and numerical deltas with respect to the short-term rate at time t . The former delta stands for the delta derived from the continuous update approach of the IDI index and the latter is the delta calculated from the numerical result developed to price the discrete updating case. The numerical delta is able to perfectly daily replicate the IDI option price in the discrete updating approach. A central finite difference scheme was used to obtain the numerical derivatives to approximate equation (32).

These last results assure the need to treat IDI index realistically when pricing and hedging options. Both strategies results in approximately normal distributed absolute hedging errors. We see that using the continuous updating approach to daily hedge the position in a call or put IDI option in a world where the index accumulates discretely leads to relative hedging errors as high as the order of the pricing errors.

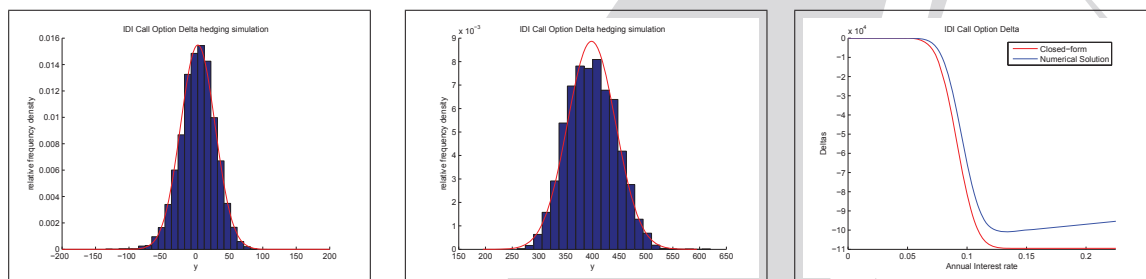


Figure 5: *Hedging IDI call option with discrete and continuous IDI updating*

$IDI(0) = 100000$, $Strike = 109500$, $a = 0.1265$ $b = 0.0802$, $\sigma = 0.0218$, $T = 252$ days

6. Conclusion and Final Discussions

We addressed the IDI Option Pricing and Hedging problem using the discretely compounded ID index hypothesis in lieu of the continuously compounded version. This models the problem realistically, according to the BMF&Bovespa standards. On the other hand we obtain an approximate value for the option price, using PDE techniques, which have its own challenges. So, comparing with the existing literature, we obtain the approximate price for the exact problem instead of obtaining the exact price for the approximate problem. However, our method can be refined approaching the benchmark, which means obtaining the exact price for exact problem.

An equally important result is that we developed a numerical scheme that is free from spurious oscillation, for any choice of drift and volatility parameters. We emphasize the fact that

the Modified Full-Implicit Finite Difference Scheme we developed is IDI-independent, thus can be extended to any interest rate derivative.

References

- Almeida, C. & Vicente, J. V. M. (2012), 'Term structure movements implicit in asian option prices', *Quantitative Finance* **12**, 119–134.
- Barbachan, J. S. F. & Ornelas, J. R. H. (2003), 'Apreçamento de opções de IDI usando o modelo CIR', *Estudos Econômicos* **33**(2), 287–323.
- Bollen, N. P. B. (1997), 'Derivatives and the price of risk', *The Journal of Futures Markets* **17**(7), 839–854.
- Brace, A. (2008), *Engineering BGM*, Financial Mathematics Series, Chapman Hall/CRC, Florida.
- Burden, R. & Faires, J. (2010), *Numerical Analysis*, Cengage Learning.
- Chan, K. C., Karolyi, G. A., Longstaff, F. A. & Sanders, A. B. (1992), 'An empirical comparison of alternative models of the short-term interest rate', *The Journal of Finance* (3), 1209–1227.
- Cox, J. C., Ingersoll, J. & Ross, S. (1986), 'A theory of the term structure of interest rates', *Econometrica* **53**, 385–407.
- Duffy, D. (2004), 'A critique of the crank-nicolson strengths and weaknesses for financial instrument pricing', *WILMOTT magazine* pp. 68–76.
- Genaro, A. D. & Avellaneda, M. (2013), Pricing interest rate derivatives under monetary policy changes, in 'Anais do XIII Encontro Brasileiro de Finanças', Rio de Janeiro, Brazil.
- Iserles, A. (2009), *A First Course in the Numerical Analysis of Differential Equations*, Cambridge Texts in Applied Mathematics, 2th edn, Cambridge University Press, Cambridge.
- Milev, M. & Tagliani, A. (2013), 'Efficient implicit scheme with positivity preserving and smoothing properties', *Journal of Computational and Applied Mathematics* (243), 1–9.
- Oksendal, B. (2007), *Stochastic Differential Equations*, 6th edn, Springer, New York.
- Plemmons, R. J. (1977), 'M-matrix characterizations', *Linear Algebra and its applications* (18), 175–188.
- Tankov, P. & Cont, R. (2003), *Financial Modelling with Jump Processes*, Financial Mathematics Series, Chapman Hall/CRC, Florida.
- Tavella, D. (2002), *Quantitative Methods in Derivatives Pricing*, John Wiley & Sons, New Jersey.
- Varah, J. M. (1975), 'A lower bound for the smallest singular value of a matrix', *Linear Algebra and its applications* (11), 3–5.
- Vasicek, O. (1977), 'An equilibrium characterization of the term structure', *Journal of Financial Economics* **5**, 177–188.
- Vieira, C. & Pereira, P. (2000), Closed form formula for the price of the options on the 1 day brazilian interfinancial deposits index IDI1, in 'Anais do XXII Congresso Brasileiro de Econometria', Vol. 2, Campinas, Brazil.
- Wilmott, P. (2006), *Paul Wilmott on Quantitative Finance*, 2th edn, John Wiley & Sons, Chichester.
- Zvan, R., Forsyth, P. & Vetzal, K. (1999), 'Discrete asian barrier options', *Journal of Computational Finance* (3), 41–68.