Inventory Location Problem with Risk Pooling Effect: A Performance Analysis Between Different Formulations

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RESUMO

Este trabalho apresenta duas formulações matemáticas para o Problema de Localização de Armazéns com Efeito de Consolidação de Estoques (Inventory Location Problem with RiskPooling Effect), que prevê reduções nos estoques de segurança dos clientes a partir da sua centralização. A primeira formulação analisa o problema a partir da perspectiva de uma cadeia de abastecimento de dois elos ligados através de uma restrição de balanço de massa, totalmente baseada em variáveis de índice duplo. A outra inclui uma variável de fluxo de índice triplo conectando os fornecedores aos clientes. Ambos os modelos são linearizados usando uma aproximação por linearização por partes, tendo em conta a natureza não linear da função de estoque de segurança, e são propostos dois métodos para dividir os segmentos. Os modelos foram analisadas em vários aspectos, e os seus desempenhos foram comparados para conjuntos de dados gerados aleatoriamente com até 20 fornecedores, 20 armazéns e 150 clientes.

PALAVRAS CHAVE. Problema de Localização de Armazéns. Efeito de Consolidação de Estoques. Linearização por Partes.

Área Principal: P&G - PO na Área de Petróleo & Gás

ABSTRACT

This work presents two mathematical formulations for the Inventory Location Problem considering the risk-pooling effect, which provides reductions in customers' safety stocks by centralizing them. The first formulation analyzes the problem from the perspective of a two-echelon supply chain being connected through a mass balance constraint, totally based on double index variables. The other includes a triple index flow variable connecting the suppliers to customers. Both models are linearized using a piecewise linearization approximation, given the nonlinear nature of the safety stock function, and two methods are proposed to divide the segments. The models were analyzed in several aspects, and its performances were compared for sets of randomly generated data with up to 20 suppliers, 20 warehouses and 150 customers.

KEYWORDS. Inventory Location Problem. Risk Pooling Effect. Piecewise Linearization.

Main Area: P&G - OR in Oil & Gas

1. Introduction

The inventory location problem can be described as the design of a production-distribution-storage system in two echelons, where a set of clients have to be supplied by vendors. The warehouses should be allocated at the intermediate level in order to consolidate customers' cycle and safety stocks.

The modeling and solution of this kind of problem have great importance from both an academic and economic perspective. One can highlight the fact that they represent a decision-making support tool that includes decisions concerning the strategic and tactical levels. This translates into more comprehensive models and solutions that are best suited to their business reality. In the oil industry, where huge amounts are spent on investments and operations, this represents an incredible potential of savings and efficiency gains in the supply chain as a whole. The uncertainties surrounding the demand are also addressed, which makes the models more adherent to reality.

From an academic point of view it is possible to point the problem complexity, given its integer and nonlinear nature. It makes necessary the application of linearization techniques that reconcile a low approximation error with the possibility to find feasible solutions with reasonable computational cost. Being a relatively new field of study, there are many opportunities for improvement on existing methods and relevant findings.

Several approaches have been proposed in the literature to deal with the problems of inventory and location, as well as their combinations and deployments. The classical inventory problems sought to minimize stocks keeping a certain service level, but the facility locations and customers allocation were prefixed, while opening and transportation costs were ignored. On the other hand, classical facility location models dealt with warehouses and transportation costs, but stocks were oversimplified. Due to their intrinsic complexity, these problems were dealt separately until few years ago, when the seminal work of Das and Tyagi (1997) was presented, followed by Barahona and Jensen (1998), Nozick and Turnquist (1998), Daskin et al. (2002), Vidyarthi et al. (2007), You and Grossmann (2008), Gebennini et al. (2009), Montebeller Junior (2009), Silva (2012), between others.

The Economic Order Quantity model (EOQ) was one of the first to be developed, proposed by Harris (1913) to optimize inventory levels when the demand is continuous and perpetual. Subsequently, the EOQ has become an integral part of the Reorder Point model (ROP) and the demands, as well as resupply lead times, became probabilistic (Ballou, 2005). The EOQ determines the batch size that minimizes the total cost, which is the sum of the order and inventory maintenance costs. The reorder point, on the other hand, is the moment when the purchase order should be done.

The uncertainties of customers demand and suppliers service level require an additional stock, which is called safety stock (SS). The SS is, in general, calculated from the probability of non-occurrence of a stock-out during the lead time. Bowersox and Closs (2001) assert that this demand is often considered adherent to the normal distribution, which is the simplest among those used in order to control stocks.

Jorge (2008) lists four subdivisions of risk pooling techniques: stocks centralization (or consolidation), order splitting, cross-filling (or transshipment) and components standardization. The stocks consolidation, also known as portfolio effect, aims to mitigate the risk arising from demand fluctuations of geographic separate customers through the sharing of distribution network units. The savings attributed to this consolidation have been studied by Maister (1976) through the square root rule[1]. Eppen (1979) deepens this study, deriving an expression of the expected cost of stock-out and stocks maintenance as a function of each warehouse demand parameters, also considering the correlation coefficient between demands.

[1] Subject to certain assumptions about the probability distribution of lead time and demand, and the lack of correlation between demands, this rule states that the centralization of n deposits in just one reduces the safety stocks proportionally to $\sqrt{n}$.

According Ambrosino et al. (2005), the facility location problem consists in determining the best way to transfer goods to demanding points, choosing the network structure while the total costs of are minimized. The first studies involving this type of decision were made for Thünen and Weber in the late nineteenth century and early twentieth century (Ballou, 2004). However, the growth of these studies occurred around the 1960s.

Hakimi (1964) proposed the p-median problem, which correspond to localize p medians in a network, aiming to minimize the sum of all distances between each demanding point and their closest facility. The fixed charge facility location problem is a variation of the p-median problem, with fixed facilities cost and no capacity restrictions.

Jayaraman (1998) extends the classic facility location problem incorporating multiple commodities and transportation modes, as well as charging the cycle stock holding and handling costs in the objective function. Ambrosino et al. (2005) included intermediate depots and also dealt with distribution routes that can attend multiples customers per trip.

The work of Das and Tyagi (1997) presents one of the first formal analysis of inventory centralization decision in a broader perspective, with a joint model of facility location and non-simplified treatment of cycle and safety stocks. Several total cost elements are represented, and their effects over the centralization are examined in five scenarios.

The models presented here are based on those proposed by Ferreira Filho and Gendron (2012), and have many similarities with the formulation of Vidyarthi et al. (2007).

2. Definitions

More formally: given a set of customers and their demands, the potential location of deposits, the available suppliers and the respective annual cost of opening and operation; the inventory location problem can be defined as determining which suppliers should be used, which deposits must be open, the clients-deposits allocation and the stocks level at each deposit; in order to minimize the opening and fixed costs of suppliers and deposits chosen, the transportation costs between them, as well as between warehouses and customers, and the stocks holding total costs. As constraints, we have the suppliers and warehouses capacity, the mass balance constraint[2], the total fulfillment of customers' demand for a given service level, and the clients covering.

2.1. Assumptions and Hypothesis

The models assume some hypothesis and prerequisites in order to be consistent with the formulations proposed:

i. Each supplier and customer has its location fixed and predetermined;
ii. Warehouses' potential locations are predetermined and finite;
iii. The number of warehouses is finite and at most equal to the number of potential locations;
iv. The demands of each customer are independent, constant over time and follow a normal distribution with a non-zero standard deviation.
v. The service level required by each client is considered as a safety factor that guarantees certain probability of non-occurrence of stock-outs;
vi. Customers should be served by one and only one warehouse, that is, you can't have cross-filling;
vii. Each allocated warehouse can serve one or more customers;
viii. The safety stock of all customers served by a particular warehouse is kept at this warehouse;
ix. Cycle and safety stocks are considered only in warehouses;
x. Inventories are not considered in the suppliers or clients;
xi. The inventory management model in warehouses is assumed to be the EOQ and ROP;
xii. The unitary storage costs are functions of the resupply lead time of deposits for each supplier;
xiii. The unitary transportation costs between suppliers, deposits and customers depends on the distance between them.

[2] The flow of commodities coming from the suppliers of each deposit must be equal the flow it forwards to attend its customers' cycle demand plus the safety stock it holds.

3. Formulations

This problem can be faced from two different perspectives. The first perspective divides it in two echelons, where the first concerns the flow of commodities from suppliers to deposits in order to fulfill its cycle and safety stocks; and the second echelon comprehends the flow from deposits to customers. These flows are equilibrated by a mass balance constraint. As this formulation contains only variables with up to two indices it will be called Double Index Formulation (DI).

The other perspective breaks the model not by its echelons, but by the purpose of the produced commodities. Commodities produced to attend the customers' cycle demand are addressed by the variable x_{jkl}, that defines the flow of products from the supplier j to the customer l passing through the deposit k; and the variable xs_{jk} determines the flow of products shipped from j in order to supply all the safety stocks consolidated in deposit k. As the variable x_{jkl} contain three indices, this will be called Triple Index Formulation (TI).

3.1. Nomenclature

Indices and Sets

- $j \in J$ — Index and set of potential suppliers to be used
- $k \in K$ — Index and set of potential deposits to be opened
- $l \in L$ — Index and set of customers
- $r \in R$ — Index and set of piecewise linearization segments

Parameters

- Γ — Planning horizon: number of productive days in one year
- μ_l — Mean of the daily demand of customer l
- σ_{il} — Covariance of demand d_i and d_l
- σ_l — Standard deviation of the daily demand of customer l
- σ_l^2 — Variance of of the daily demand of customer l
- φ_{kl} — Transportation lead time between the deposit k and the customer l
- ξ_{kl} — Customers coverage: 1 if the customer l can be served by the deposit k, 0 otherwise
- c_{jk} — Unitary production and shipment cost of a product from the supplier j to deposit k
- d_l — Demand distribution of customer l
- f_j — Annual fixed operational cost of a supplier at the location j
- g_k — Annual fixed opening and operational cost of a deposit at the location k
- h_k — Annual unitary stocks keeping and handling cost at deposit k
- t_{kl} — Unitary shipment cost between the deposit k and the customer l
- Z_α — Safety factor correspondent to the desired service level α
- C_k^r — Linear coefficient of segment r in the piecewise linearization of the SS_k function
- F_k^r — Linear coefficient of segment r in the piecewise linearization of the SS_k function
- L_k^r — Upper bound of segment r abscissa in the piecewise linearization of the SS_k function

Variables

- SS_k — Safety stock consolidated at the deposit k
- w_j — Choice of suppliers: 1 if the supplier j is used, 0 otherwise
- x_{jk} — Number of unities produced at the supplier j and sent to deposit k (double index model)
- y_{kl} — Customers allocation: 1 if customer l is allocated to deposit k, 0 otherwise
- z_k — Deposits opening: 1 if the deposit k is used, 0 otherwise
- u_k^r — Binary auxiliary variables of piecewise linearization: 1 if segment r is active in the SS approximation of deposit k, 0 otherwise.
- v_{kl}^r — Continuous auxiliary variables of

piecewise linearization which determines the abscissa of the SS of customer l allocated in deposit k when segment r is active

3.2. Double Index Basic Formulation

As mentioned above, Eppen (1979) demonstrate an expression of the expected cost of stock-out and stocks maintenance at one warehouse. The derivative of this expression give us that the optimal service level is $\alpha = \frac{c_f}{c_f+c_e}$, where c_f is stock-out cost and c_e is the stock excess cost. Let Z_α be the α^{th} fractile point of the demand distribution d and assuming $d \sim N(\mu, \sigma^2)$, we have that $\mu + Z_\alpha\sigma$ is the minimum amount of products we must take in stock to attend our customers α percent of the time.

Considering that each deposit k holds the stock of only one customer l whose lead-time φ_{kl} is deterministic, the demand of l during the lead time would be $d_l \sim N(\varphi_{kl}\mu_l, \varphi_{kl}\sigma_l^2)$. Thus, the stock at deposit k must be $\varphi_{kl}\mu_l + Z_\alpha\varphi_{kl}\sigma_l$, being $\varphi_{kl}\mu_l$ the cycle stock and $Z_\alpha\varphi_{kl}\sigma_l$ the safety stock. In this scenario, the total safety stock of the network is given by

$$SS_{Total}^{Decentralized} = \sum_{k\in K} SS_k^{Decentralized} = \sum_{k\in K}\sum_{l\in L} Z_\alpha\varphi_{kl}\sigma_l y_{kl}$$

where y_{kl} is the binary variable that determines the allocation of customers to deposits.

Beyond, considering the case that the demands of all customers are consolidated in one deposit T, $d_T = \sum_{l\in L} d_l$. Thus, $d_T \sim N(\mu_T, \sigma_T^2)$, where $\mu_T = \sum_{l\in L}\varphi_{Tl}\mu_l$ and $\sigma_T^2 = \sqrt{\sum_{i\in L}\sum_{l\in L}\varphi_{Tl}\sigma_{il}}$. For simplification, the demands are considered uncorrelated, which results in $\sigma_T^2 = \sqrt{\sum_{l\in L}\varphi_{Tl}\sigma_l^2}$ and consequently $SS_T = Z_\alpha\sqrt{\sum_{l\in L}\varphi_{Tl}\sigma_l^2}$. Expanding the argument to any intermediate deposits consolidation, we have

$$SS_{Total}^{Centralized} = \sum_{k\in K} SS_k^{Centralized} = \sum_{k\in K} Z_\alpha\sqrt{\sum_{l\in L}\varphi_{kl}\sigma_l^2 y_{kl}}$$

Starting from this point, the non-linear basic formulation below can be stated.

$$\text{Minimize} \quad \sum_{j\in J} f_j w_j + \sum_{j\in J}\sum_{k\in K}(c_{jk}+h_k)x_{jk} + \sum_{k\in K} g_k z_k + \Gamma\sum_{k\in K}\sum_{l\in L} t_{kl}\mu_l y_{kl} \quad (1)$$

$$\text{Subject to} \quad SS_k = Z_\alpha\sqrt{\sum_{l\in L}\varphi_{kl}\sigma_l^2 y_{kl}} \qquad \forall k\in K \quad (2)$$

$$\sum_{j\in J} x_{jk} = \Gamma\sum_{l\in L}\mu_l y_{kl} + SS_k \qquad \forall k\in K \quad (3)$$

$$\sum_{k\in K} x_{jk} \le P_j w_j \qquad \forall j\in J \quad (4)$$

$$\Gamma\sum_{l\in L}\mu_l y_{kl} + SS_k \le V_k z_k \qquad \forall k\in K \quad (5)$$

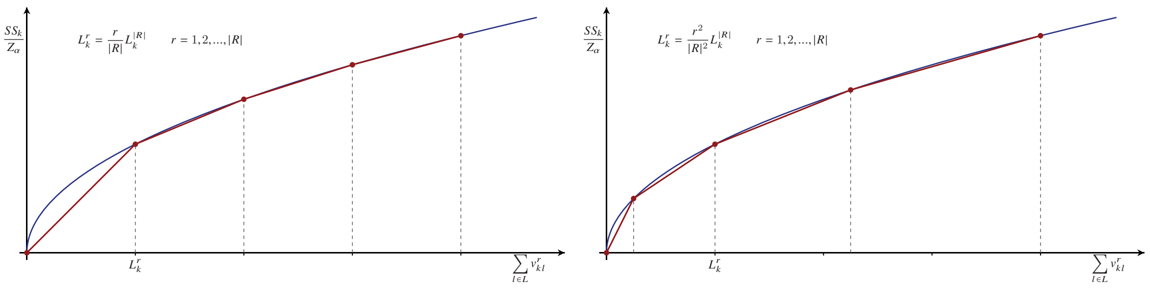
$$\sum_{k\in K} y_{kl} = 1 \qquad \forall l\in L \quad (6)$$

$$y_{kl} \le \xi_{kl} z_k \qquad \forall l\in L, \forall k\in K \quad (7)$$

$$x_{jk}, SS_k \ge 0 \qquad \forall j\in J, \forall k\in K \quad (8)$$

$$w_j, z_k \in \{0,1\} \qquad \forall j\in J, \forall k\in K \quad (9)$$

$$y_{kl} \in \{0,1\} \qquad \forall l\in L, \forall k\in K \quad (10)$$



(a) Linear segmentation (LS) (b) Quadratic segmentation (QS)

Figure 1: Segmentations of the piecewise linearization approximation function

The objective function (1) minimizes the sum of opening and fixed costs of suppliers and warehouses (amortized annually), production and transportation costs from suppliers to deposits, inventory handling and maintenance costs of cycle and safety stocks in warehouses, and transportation costs from warehouses to customers.

Constraint (2) calculate safety stock for each opened warehouse. Constraint (3) determines the products flow conservation in the warehouses. Constraints (4) and (5) limit the capacity of suppliers and warehouses, and allow only those who are open to be used. Constraint (6) provides customers allocation to deposits. Given the binary nature of variable y_{kl} it also prevents cross-filling. Constraint (7) limits customers covering, preventing their allocation to deposits that can't meet them[3]. The constraints (8), (9) and (10) satisfy the integrality requirements and non-negativity of the variables.

3.3. Piecewise Linearization

Given the non-linear nature of equation (2), it's impossible to solve the original model with the classical optimization methods tailored to deal with linear problems. Thus, it's necessary to use some linearization procedure to approximate this by a linear function. Using the piecewise linearization approach proposed by Croxton and Gendron (2003), the function $Z_\alpha \sqrt{\sum_{l \in L} \varphi_{kl} \sigma_l^2 y_{kl}}$ is replaced for $Z_\alpha \sum_{r \in R} \left(F_k^r u_k^r + C_k^r v_k^r \right)$, divided in r segments. F_k^r is the linear coefficient of each segment r of each deposit k, and C_k^r is the slope. The binary variable u_k^r determines which segment is active in the approximation, and the abscissa is originally defined by v_k^r. In order to get a stronger formulation, the v_k^r variable can be disaggregated into $\sum_{l \in L} v_{kl}^r$, resulting in the equation below.

$$SS_k = Z_\alpha \sum_{r \in R} \left(F_k^r u_k^r + C_k^r \sum_{l \in L} v_{kl}^r \right)$$

Figure 1 shows two graphical representations of this procedure, with different segmentation approaches. The first (subfigure 1a), most intuitive and common in the literature (Ferreira Filho and Gendron, 2012; Croxton and Gendron, 2003; Croxton et al., 2007; Frangioni and Gendron, 2009), consists of breaking the linearized function in segments equally spaced by the abscissas axis. The limitants L_k^r are directly proportionals to the r segment index, as one can see in the formula below. Due to its linear relation to r, this approach will be called linear segmentation (LS).

$$L_k^r = \frac{r}{|R|} L_k^{|R|} = \frac{r}{|R|} \max \sum_{l \in L} v_{kl}^r \qquad r = 0, 1, \ldots, |R| \quad \forall k \in K$$

[3] By default, ξ_{kl} was considered 1 in all experiments realized.

The upper bound $L_k^{|R|} = \max \sum_{l \in L} v_{kl}^r$ was primarily considered the same for all the deposits, and was estimated as

$$L_k^{|R|} = \left\lceil \sum_{l \in L} \left(\max_k \varphi_{kl} \right) \sigma_l^2 \right\rceil$$

The second linearization approach, shown in subfigure 1b, will be called quadratic segmentation (QS). It uses the inverse function of the safety stock (a square root function) in order to determine the segmentation. Thus, the L_k^r limitants are proportional to the square of the index of the segment r, using the same upper bound $L_k^{|R|}$ defined above.

$$L_k^r = \frac{r^2}{|R|^2} L_k^{|R|} = \frac{r^2}{|R|^2} \max \sum_{l \in L} v_{kl}^r \qquad r = 0, 1, ..., |R| \quad \forall k \in K$$

Three alternative configurations were also tested[4] to be compared with SL, and the SQ was chosen for been the one who minimizes the difference between the area under the SS_k curve and the area delimited by the segments, determined through the definite integrals. Therefore, it's expected that this approach diminishes the average linearization error.

3.4. Double Index Complete and Linearized Formulation

Frangioni and Gendron (2009) and Ferreira Filho and Gendron (2012) use some redundant constraints in order to strengthen the formulation, limiting better the convex hull of the relaxed model. The double index complete formulation is shown below, linearized by the piecewise linearization approach.

Minimize
$$\sum_{j \in J} f_j w_j + \sum_{j \in J} \sum_{k \in K} (c_{jk} + h_k) x_{jk} + \sum_{k \in K} g_k z_k + \Gamma \sum_{k \in K} \sum_{l \in L} t_{kl} \mu_l y_{kl} \tag{1}$$

Subject to
$$SS_k = Z_\alpha \sum_{r \in R} \left(F_k^r u_k^r + C_k^r \sum_{l \in L} v_{kl}^r \right) \qquad \forall k \in K \tag{2a}$$

$$\sum_{l \in L} \varphi_{kl} \sigma_l^2 y_{kl} = \sum_{r \in R} \sum_{l \in L} v_{kl}^r \qquad \forall k \in K \tag{11}$$

$$L_k^{r-1} u_k^r \leq \sum_{l \in L} v_{kl}^r \leq L_k^r u_k^r \qquad \forall k \in K, \forall r \in R \tag{12}$$

$$\sum_{r \in R} u_k^r \leq 1 \qquad \forall k \in K \tag{13}$$

$$v_{kl}^r \leq \varphi_{kl} \sigma_l^2 u_k^r \qquad \forall k \in K, \forall l \in L, \forall r \in R \tag{14}$$

$$\sum_{j \in J} x_{jk} = \Gamma \sum_{l \in L} \mu_l y_{kl} + SS_k \qquad \forall k \in K \tag{3}$$

$$\sum_{k \in K} x_{jk} \leq P_j w_j \qquad \forall j \in J \tag{4}$$

$$x_{jk} \leq P_j w_j \qquad \forall j \in J, \forall k \in K \tag{15}$$

$$\sum_{j \in J} \sum_{k \in K} x_{jk} \geq \Gamma \sum_{l \in L} \mu_l \tag{16}$$

$$\sum_{j \in J} \sum_{k \in K} x_{jk} \leq \Gamma \sum_{l \in L} \mu_l + Z_\alpha \left\{ \sum_{k \in K} \left(\max_{r,k} F_k^r \right) + \left(\max_{r,k} C_k^r \right) \sum_{l \in L} \varphi_{kl} \sigma_l^2 \right\} \tag{17}$$

[4]A cubic segmentation, a logarithmic and even a square root segmentation.

$$\Gamma \sum_{l \in L} \mu_l y_{kl} + SS_k \leq V_k z_k \qquad \forall k \in K \quad (5)$$

$$\sum_{j \in J} x_{jk} \leq V_k z_k \qquad \forall k \in K \quad (18)$$

$$\sum_{k \in K} y_{kl} = 1 \qquad \forall l \in L \quad (6)$$

$$y_{kl} \leq \xi_{kl} z_k \qquad \forall l \in L, \forall k \in K \quad (7)$$

$$x_{jk}, SS_k, v_{kl}^r \geq 0 \qquad \forall j \in J, \forall k \in K, \forall l \in L, \forall r \in R \quad (8a)$$

$$w_j, z_k, u_k^r \in \{0,1\} \qquad \forall j \in J, \forall k \in K, \forall r \in R \quad (9a)$$

$$y_{kl} \in \{0,1\} \qquad \forall l \in L, \forall k \in K \quad (10)$$

Constraints (2a), (11), (12) and (13) are responsible for the piecewise linearization. The redundant constraints (14), (15), (16), (17) and (18) are used to strengthen the formulation.

3.5. Triple Index Complete and Linearized Formulation

Minimize
$$\sum_{j \in J} f_j w_j + \sum_{j \in J} \sum_{k \in K} \sum_{l \in L} (c_{jk} + t_{kl}) x_{jkl} + \sum_{k \in K} g_k z_k + \sum_{j \in J} \sum_{k \in K} c_{jk} xs_{jk} + \sum_{k \in K} h_k \left(\sum_{j \in J} \sum_{l \in L} x_{jkl} + \sum_{j \in J} xs_{jk} \right) \quad (19)$$

Subject to
$$SS_k = Z_\alpha \sum_{r \in R} \left(F_k^r u_k^r + C_k^r \sum_{l \in L} v_{kl}^r \right) \qquad \forall k \in K \quad (2a)$$

$$\sum_{l \in L} \varphi_{kl} \sigma_l^2 y_{kl} = \sum_{r \in R} \sum_{l \in L} v_{kl}^r \qquad \forall k \in K \quad (11)$$

$$L_k^{r-1} u_k^r \leq \sum_{l \in L} v_{kl}^r \leq L_k^r u_k^r \qquad \forall k \in K, \forall r \in R \quad (12)$$

$$\sum_{r \in R} u_k^r \leq 1 \qquad \forall k \in K \quad (13)$$

$$v_{kl}^r \leq \varphi_{kl} \sigma_l^2 u_k^r \qquad \forall k \in K, \forall l \in L, \forall r \in R \quad (14)$$

$$\sum_{j \in J} x_{jkl} = \Gamma \mu_l y_{kl} \qquad \forall k \in K, \forall l \in L \quad (20)$$

$$\Gamma \sum_{l \in L} \mu_l y_{kl} + SS_k \leq V_k z_k \qquad \forall k \in K \quad (5)$$

$$\sum_{k \in K} \sum_{l \in L} x_{jkl} + \sum_{k \in K} xs_{jk} \leq P_j w_j \qquad \forall j \in J \quad (21)$$

$$\sum_{k \in K} \sum_{l \in L} x_{jkl} + \sum_{k \in K} xs_{jk} \leq V_k z_k \qquad \forall k \in K \quad (22)$$

$$\sum_{j \in J} xs_{jk} = SS_k \qquad \forall k \in K \quad (23)$$

$$\sum_{k \in K} y_{kl} = 1 \qquad \forall l \in L \quad (6)$$

$$y_{kl} \leq \xi_{kl} z_k \qquad \forall l \in L, \forall k \in K \quad (7)$$

$$x_{jkl}, xs_{jk}, SS_k, v^r_{kl} \geq 0 \qquad \forall j \in J, \forall k \in K \quad (8b)$$

$$w_j, z_k, u^r_k \in \{0,1\} \qquad \forall j \in J, \forall k \in K \quad (9a)$$

$$y_{kl} \in \{0,1\} \qquad \forall l \in L, \forall k \in K \quad (10)$$

Like in the double index formulation, the objective function (19) aims to minimize the total opening, logistcs and operational costs of the supply chain. Constraints (2a), (11), (12) and (13) linearize the safety stock function. Constraint (20) assigns the suppliers production to attend the cycle demand of customers, while (5) and (21) limit the capacity of suppliers and deposits, respectively. Finally, constraint (23) assign the safety stock to the variable xs_{jk}. The redundant constraints (14) and (22) strengthen the formulation; and the others (6, 7, 8b, 9a, 10) were already shown.

4. Experiments and Analysis

IBM® ILOG CPLEX Optimizer v12.4 was chosen as the solver, opting for the simplicity of OPL (Optimization Programming Language) to build the models. No pre-processing, cuts or heuristics were used, as well as any other *ad hoc* method. The communication interfaces were created using the Concert layer integration with C ++. The instances generation method was implemented using VBA (Visual Basic for Applications) and Microsoft® Excel 2010[5].

The experiments were performed on 22 computers belonging to the Amazon® Elastic Compute Cloud (Amazon EC2). Every unit was a c3.large instance with two 64-bit vCPUs, 3.75 GB RAM and running Microsoft® Windows Server 2012. Each vCPU is a hardware hyperthread of Intel® Xeon® processors E5-2680v2 2.8 GHz Ivy Bridge (Amazon Web Services, 2014).

4.1. Test Instances

Each instance receives as input the number of plants, warehouses and customers; the utilization coefficient of suppliers (UCS) and utilization coefficient of deposits (UCD); and the number of segments the SS function should be divided.

The UCS is utilized to calculate the capacity of each supplier and can be seen as the expected value of UFS (Utilization Factor of Suppliers). Analogously, the UCD can be seen as the expected value of the UFD (Utilization Factor of Deposits). The UFS and UFD, in turn, are respectively given by

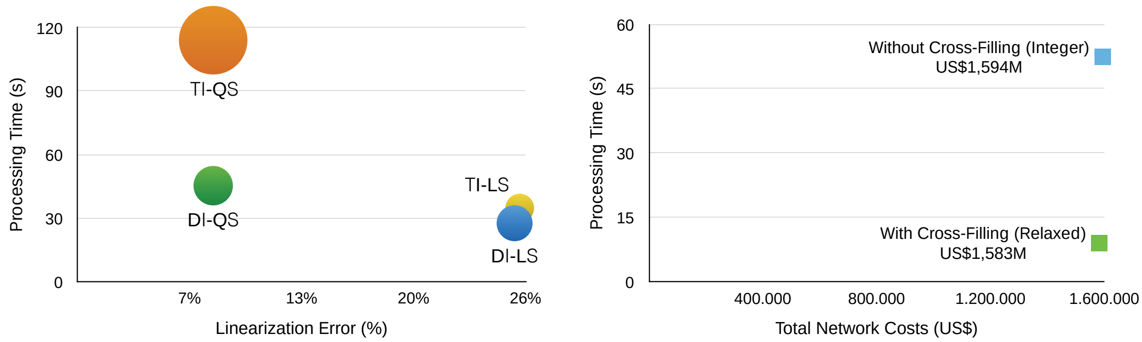
$$\frac{\Gamma \sum_{l \in L} \mu_l + \sum_{l \in L} \left(\max_k \varphi_{kl} \right) \sigma_l^2}{\sum_{j \in J} P_j} \quad \text{and} \quad \frac{\Gamma \sum_{l \in L} \mu_l + \sum_{l \in L} \left(\max_k \varphi_{kl} \right) \sigma_l^2}{\sum_{k \in K} V_k}$$

The generation method was inspired in Ferreira Filho and Gendron (2012), which was, by its time, based on Vidyarthi et al. (2007) proposal. However, some modifications had to be introduced in order to allow the following experiments. The complete process is detailed in Bittencourt (2014).

4.2. Experiments

Four submodels were tested: DI-LS (double index model with linear segmentation), DI-QS (double index model with quadratic segmentation), TI-LS (triple index model with linear segmentation) and TI-QS (triple index model with quadratic segmentation). The examples were created based on data of the 150 greatest continental USA cities, according to 2000 census, containing up to 20 suppliers, 20 deposits and 150 customers.

[5]The OPL and C++ code, as well as the instances generator, used datasets and results are publicly available in a repository hosted at `http://bit.ly/InventoryLocation`



(a) Processing Time x Linearization Error x Standard Deviation

(b) Cross-Filling Influence in Processing Time and Total Cost

Figure 2: Random instances experiments

4.2.1. Random

The first data-set was composed by ten totally random instances, totaling 40 results. Thirteen instances were generated, but three of then were unfeasible or didn't find the optimal solution within 7.200 seconds.

The results of the first experiment are shown in graph 2a where abscissas axis represent the average linearization error of the safety stock function, the ordinate show the average processing time, and the size of the bubbles represent the standard deviation of the processing time. As one can see, the DI-QS model showed the most balanced result between processing time, consistence and linearization error.

4.2.2. Cross-Filling

Using the same data-set of the first experiment, the second run aimed to investigate the effect of relaxing the integrality requirement of y_{kl}, that is, to allow the cross-filling policy. To enable this condition permits a customer to be attended by more than one deposit, which tends to diminish the total operational cost of the network. While the impact in costs wasn't relevant, keeping the same solution of the binary y_{kl} in most cases, the impact in processing time was very significant, as shown in subfigure 2b.

4.2.3. Number of segments

The third experiment was built to verify the influence of the number of segments over the computational cost and the linearization error. There were generated ten instances with 10 suppliers, 15 deposits, 50 customers, UCS and UCD of 50%. The results are presented in table 1.

Table 1: Influence of the number of segments over processing time and linearization error.

# of Segments	Average Processing Time (s)				Average Linearization Error $(\%)$			
	DI-LS	DI-QS	TI-LS	TI-QS	DI-LS	DI-QS	TI-LS	TI-QS
5	19.6	27.4	71.5	44.8	59.2%	26.4%	59.2%	26.5%
7	36.6	33.0	124.7	62.9	52.7%	16.3%	52.6%	16.1%
10	45.3	184.6	129.5	699.7	44.4%	9.0%	45.0%	8.9%
13	47.8	279.4	93.8	804.1	39.7%	5.4%	39.7%	5.4%
16	70.9	1791.5	120.2	944.9	33.5%	3.4%	33.5%	3.4%
20	222.0	2433.6	513.6	1402.6	28.7%	2.2%	28.6%	2.2%

Although the processing times for SQ models were significantly higher than SL for almost all experiments with the same number of segments, the difference in quality of the obtained solutions also was significant.

The average linearization error of SL models with 20 segments was higher than the error encountered with SQ segmentation into 5 parts. Nevertheless, DI-LS average processing time for 20 segments was eight times higher than DI-QS with 5 and more than eleven times higher comparing TI-LS with TI-QS.

5. Conclusions

This work presented two distinct formulations to deal with the Inventory Location Problem, both considering the risk pooling effect. Due to the nonlinear nature of the safety stock function, it had to be linearized so that the problem could be handled effectively. Therefore, two different segmentation were proposed, being one with equally segmented (linear segmentation) and the other proportional to the square of the segment index (quadratic segmentation).

Experiments showed that the upper bound $L_k^{|R|}$ used for segmentation was overestimated, resulting in significant linearization errors. However, the SQ linearization SQ revealed good results in practicable computational time, especially when used with the double index model (DI-QS).

Increasing the number of sections is a powerful tool to reduce the linearization error, but it must be used thoughtfully, given the impact it has on the computational cost. In quadratic segmentation, it was possible to decrease the initial error from 26.4% to 2.2%, but with a substantial growth of time.

Still on the quadratic segmentation, it was found that the double index model (ID) performed better when the number of sections was shorter (up to 13 segments), while the triple index model (IT) became more attractive when addressing a larger number of segments.

This research can raise in several ways. More immediately, clever ways to define the upper bound of segmentation and an iterative convergent linearization method will be tested. As future work, these models will be enriched through a dynamic perspective, with scenarios that vary over time; and providing multiple commodities. We also intent to use advanced solution methods, such as Lagrangian relaxation in conjunction with rows and columns generation.

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