A Computational Study of the Adjacency, Distance and Laplacian Spectra of Twin Graphs

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ABSTRACT

A twin graph is a minimum-size 2-geodetically connected graph that is not the cube graph Q_3 nor the complete graph K_3. Those graphs can be obtained from the cycle C_4 by a recursive algorithm and there are well defined operations between trees and twin graphs. All twin graphs of the same order have the same number of edges and, therefore, the same average degree. They are planar, and also free of triangles, since their faces are all isomorphic to the cycle C_4. In this paper we study computationally the adjacency, distance and Laplacian spectra of twin graphs of order 4 up to 14, analyse and compare them to the respective spectra of trees of the same order.

KEYWORDS. Twin Graphs. Graph Theory. Graph Spectra.

Theory and Algorithms in Graphs.

1. Introduction

Given a finite set A let $\mathscr{P}(A)$ denote the power set of A and let $\mathscr{P}_2(A) := \{\{x,y\} \in \mathscr{P}(A) \mid x, y \in A \text{ and } x \neq y\}$. By vacuity, when $A = \emptyset$ or when A is a singleton we define $\mathscr{P}_2(A) = \emptyset$. A *finite simple graph* with *vertices* in a set V and *edges* in a set E is a mathematical structure (V, E), where $\mathrm{E} \subset \mathscr{P}_2(\mathrm{V})$. Consider $n \in \mathbf{N}$ and let $\mathrm{G}_n := \{(\mathrm{V},\mathrm{E}) \mid |\,\mathrm{V}\,| = n \text{ and } \mathrm{E} \subset \mathscr{P}_2(\mathrm{V})\}$ denote the set of all finite simple graphs of n vertices. Given a graph $X \in \mathrm{G}_n$ we denote by $\mathrm{V}(X)$ and $\mathrm{E}(X)$ its vertex and edge sets, respectively, and by $\deg v$ the degree of the vertex $v \in \mathrm{V}(X)$.

For this section, let $X \in \mathrm{G}_n$ be a connected graph. The *open neighbourhood* of a vertex $u \in \mathrm{V}(X)$, denoted as $\mathrm{N}(u)$, is the set of vertices adjacent to u in X. A *path* in X is an induced subgraph of X that is isomorphic to a path graph. The graph X is said to be *connected* when for every $u, v \in \mathrm{V}(X)$ there is a path that connects u and v, i.e. u and v are vertices of the same induced subgraph of X that is isomorphic to a path graph, for every $u, v \in \mathrm{V}(X)$.

Given $u, v \in \mathrm{V}(X)$, a u-v *geodesic* is a path in X that connects the vertices u and v using the smallest possible number of edges. In other words, a geodesic in a graph is a shortest path between two vertices. Using this concept, one can define the *distance* between the vertices u and v as the number of edges of a geodesic that connects u and v.

Define a bijective labelling $\rho : \{1, \ldots, n\} \longrightarrow \mathrm{V}(X)$ given by $\rho(i) = v_i$. The *distance matrix* of X is a square matrix $\mathbf{d}_X(i,j) = d_X(v_i, v_j)$ of order n, where $d_X(v_i, v_j)$ represents the distance between the vertices v_i and v_j in X.

We shall represent the *spectrum* of a square matrix $\mathbf{m}$ of order n over $\mathbf{C}$ by the set $\Lambda(\mathbf{m}) := \{\alpha \in \mathbf{C} \mid \det(\alpha\, \mathbf{id}_n - \mathbf{m}) = 0\}$. Since the adjacency, Laplacian and distance matrices of a finite simple graph are real symmetric matrices, it follows that in our studies $\Lambda \subset \mathbf{R}$ and, therefore, the order relation $\geq$ of $\mathbf{R}$ enables us to consider the ordered spectra of these matrices. In this case, we will often refer to the maximum element of Λ as the *index* of the studied matrix, e.g., index of the Laplacian matrix. Also, we shall represent the eigenvalues of the adjacency, distance and Laplacian matrices by λ, ∂ and μ, respectively.

A well-known spectral metric in graph theory is the *distance energy* of a connected graph [Indulal et al., 2008]. The distance energy of X is given by:

$$\mathrm{E}_D(X) := \sum_{\partial \in \Lambda(\mathbf{d}_X)} |\partial|. \tag{1}$$

An important metric of a connected graph that is obtained by its distance matrix (or equivalently, by the distances in the same graph) is the *Wiener index* [Wiener, 1947]. For each vertex v of $\mathrm{V}(X)$ define $\mathrm{Tr}(v) = \sum_{u \in \mathrm{V(X)}} d_X(v,u)$ the *transmission* of v [Plesník, 1984]. The Wiener index of X is given by:

$$\mathrm{W}(X) := \frac{1}{2} \sum_{v \in \mathrm{V}(X)} \mathrm{Tr}(v). \tag{2}$$

There is a special class of graphs, known as k*-geodetically connected graphs* (abbrv. k-GC), that is closely related to the concept of distances in graphs. A connected graph is said to be a k-GC graph when the removal of at least k vertices is required to increase the distance between any pair of non-adjacent vertices [Entringer et al., 1977]. Also, an important result on the characterization of such graphs can be found in that paper, which is a version of the Menger's theorem for k-GC graphs. We state this result here as the Lemma 1. Among the k-GC graphs ($k \geq 2$) there are the minimum-size graphs, for which the removal of any edge will lead to a non-k-GC graph.

Lemma 1. [Entringer et al., 1977] Let $X \in \mathrm{G}_n$ be connected. The minimum number of vertices whose removal increases the distance between two non-adjacent vertices $u, v \in V(X)$ is the maximum number of disjoint u-v geodesics.

One operation in graph theory, among many others, is the vertex deletion of a graph and it does exactly what its name states. Given a vertex $v \in \mathrm{V}(X)$ let us denote by $X \setminus v$ the graph obtained from X by deleting its vertex v. In Lemma 2, we provide an alternative characterization of the k-GC graphs, $k \geq 2$, using this operation and the concepts of transmission and Wiener index. Figure 1 illustrates the ideas shown in this lemma for a 2-GC graph.

Lemma 2. Let $X \in \mathrm{G}_n$ be a k-GC graph, $k \geq 2$, and consider $v \in \mathrm{V}(X)$. Then,

$$\mathrm{W}(X \setminus v) = \mathrm{W}(X) - \mathrm{Tr}(v). \tag{3}$$

Proof. Since $X \in G_n$ is a k-GC graph, $k \geq 2$, there are at least k disjoint geodesics connecting each pair of non-adjacent vertices in X. Thus, the removal of a single vertex v will not change the distance between any pair of vertices $u, w \neq v$. Therefore:

$$\begin{aligned}
\mathrm{W}(X) &= \frac{1}{2} \sum_{u \in \mathrm{V}(X \setminus v)} \mathrm{Tr}(u) + \frac{1}{2}\mathrm{Tr}(v) \\
&= \frac{1}{2} \sum_{u \in \mathrm{V}(X \setminus v)} \left[\sum_{w \in \mathrm{V}(X \setminus v)} \mathrm{d}_X(u,w) + \mathrm{d}_X(u,v) \right] + \frac{1}{2} \left[\sum_{w \in \mathrm{V}(X \setminus v)} \mathrm{d}_X(v,w) \right] \\
&= \frac{1}{2} \sum_{u \in \mathrm{V}(X \setminus v)} \left[\sum_{w \in \mathrm{V}(X \setminus v)} \mathrm{d}_X(u,w) \right] + \frac{1}{2} \sum_{u \in \mathrm{V}(X \setminus v)} \mathrm{d}_X(u,v) + \frac{1}{2} \sum_{w \in \mathrm{V}(X \setminus v)} \mathrm{d}_X(v,w) \\
&= \frac{1}{2} \sum_{u \in \mathrm{V}(X \setminus v)} \left[\sum_{w \in \mathrm{V}(X \setminus v)} \mathrm{d}_X(u,w) \right] + \sum_{u \in \mathrm{V}(X \setminus v)} \mathrm{d}_X(u,v) \\
&= \mathrm{W}(X \setminus v) + \mathrm{Tr}(v).
\end{aligned}$$

□

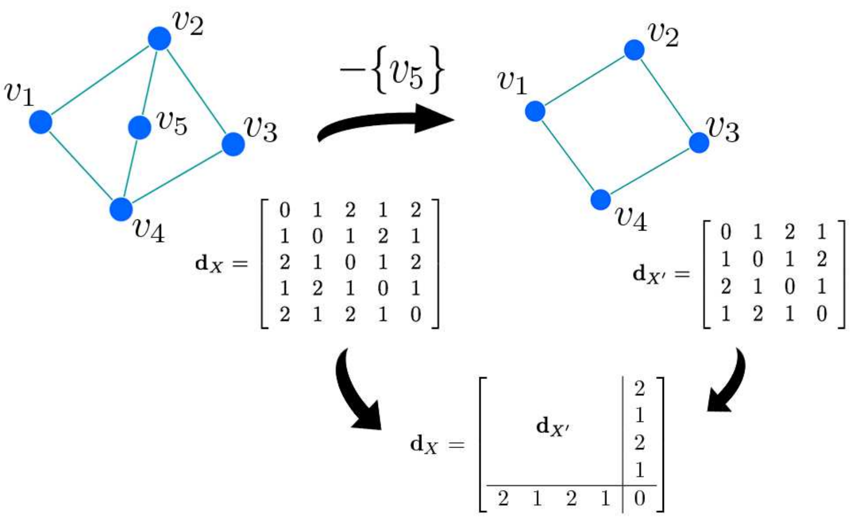


Figure 1: For a 2-GC graph X and $v \in \mathrm{V}(X)$ let $X' = X \setminus v$. The matrix $\mathbf{d}_{X'}$ is a principal submatrix of the matrix $\mathbf{d}_X$.

In this paper, we are interested in the class of twin graphs. The twin graphs have been recursively defined in [Farley and Proskurowski, 1997], as shown in Definition 1, using the concept of *twin pairs*. A *twin pair* is a pair of vertices with the same open neighbourhood [Farley and Proskurowski, 1997].

Definition 1. [Farley and Proskurowski, 1997] The cycle C_4 is a *twin graph*. Let X be a twin graph. The graph obtained by connecting a new vertex to a twin pair in X using two new edges is also a twin graph.

Interestingly, it follows by Definition 1 that all twin graphs are minimum-size 2-GC graphs, as proved in [Farley and Proskurowski, 1997]. (Notice that a different terminology is used in that paper, i.e., the 2-GC graphs are called self-repairing graphs.) Furthermore, that paper provides a fully characterization of the minimum-size 2-GC graphs. More precisely, Farley and Proskurowski (1997) proved that the class of minimum-size 2-GC graphs of order $n \geq 4$ is comprised of only the twin graphs, together with the cube graph Q_3. Notice however that the complete graph K_3 is also a minimum-size 2-GC graph.

The motivation for the name twin graph is the existence of twin pairs. Since the cube graph Q_3 is a minimum-size 2-GC but has no twin pairs, it is not a twin graph. It is important to notice that the existence of twin pairs does not make a graph a twin graph, since that graph may not be a minimum-size 2-GC graph. We shall represent the set of twin graphs of order n by $\amalg_n$.

By the definition found in [Farley and Proskurowski, 1997] it follows that every twin graph is bipartite and planar with every face isomorphic to a cycle C_4. Also, the size of the graphs in $\amalg_n$ is $2n - 4$ and their mean degree is $4 - (8/n)$.

The remaining of this paper is organized as follows. In Section 2, we study some relations between trees and twin graphs. A computational study of the spectra of twin graphs from 4 up to 14 vertices is presented in Section 3, where we analyse the spectral behavior of the twin graphs, comparing it to the spectral behavior of the trees. In Section 4 we present our conclusions and future research directions.

2. Some relations between trees and twin graphs

There is a mapping between trees and k-GC graphs for any $k \geq 2$, as observed in [Farley and Proskurowski, 1997] for twin graphs, and in [Nieminen and Peltola, 1998] for k-GC graphs in general. That transformation of trees in k-GC graphs consists of replacing the vertices in the trees by complete graphs K_p, or their complements $\overline{K_p}$, where $p \geq k$, and cross-connecting the vertices of these new added graphs, as exemplified in Figure 2.

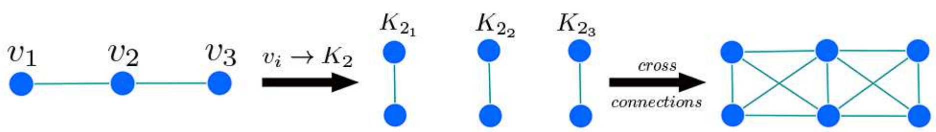


Figure 2: A transformation that leads a path graph P_3 into a 2-GC graph, which is not a twin graph since it is not minimal with respect to the number of edges.

However, by simply choosing the appropriate graphs to replace the vertices of the trees, one can always obtain a twin graph from a chosen tree. The rules for obtaining a twin graph from this transformation are given in Proposition 1. Let $\mathcal{T}_n \subset \mathrm{G}_n$ be the set of all trees of n vertices.

Proposition 1. Let $n \geq 3$ and define the function $\mathrm{T} : \mathcal{T}_n \rightarrow \bigcup_{r \geq n} \mathrm{G}_r$. Given a tree $X \in \mathcal{T}_n$, construct $\mathrm{T}(X)$ according to the following steps:

1. If $v \in \mathrm{V}(X)$ is such that $\deg v = 1$, then do nothing to v.
2. If $v \in \mathrm{V}(X)$ is such that $\deg v \geq 2$, then replace the vertex v by $\overline{K_2}$.
3. After all replacements, connect $\overline{K_{2_u}}$ to $\overline{K_{2_v}}$ if and only if u and v are adjacent in X.

Then, $\mathrm{T}(X)$ is a twin graph with N nodes and $N = \sum_{v \in \mathrm{V}(X)} \min\{\deg v, 2\}$.

Using $X = P_4$, Figure 3 exemplifies how to apply the transformation T described in Proposition 1 for obtaining a twin graph from a tree X.

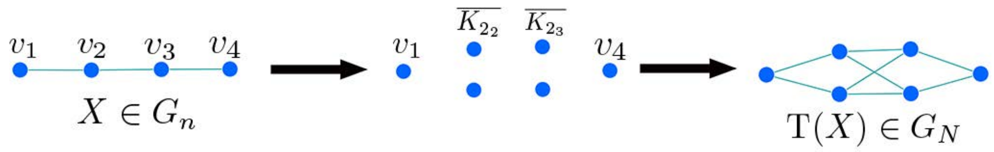


Figure 3: Example of application of the transformation T as described in Proposition 1, for $X = P_4$. A twin graph is always generated by following these steps. It is easy to verify that the number of vertices N and the number of edges M from $\mathrm{T}(X)$ are such that $M = 2(N-2)$.

The function T described in Proposition 1 allows us to generate all twin graphs of a given number of vertices, considering the correct selection of the set of trees that are going to be transformed, as shown in Corollary 1.

Corollary 1. A graph $X' \in \mathbb{I}_N$ can only be generated by the function T described in Proposition 1 if the number of vertices n of the chosen graph $X \in \mathcal{T}_n$ is such that $\lceil \frac{N}{2} \rceil + 1 \leq n \leq N - 1$.

It is easy to demonstrate Corollary 1 if the extremal tree graphs with respect to the number of replaced vertices in the mapping T are analyzed, as done next and illustrated in Figure 4. The graph of order n that has the smallest number of vertices replaced is the star graph $S_n \in \mathcal{T}_n$ and the graph with the largest number of replaced vertices is the path graph $P_n \in \mathcal{T}_n$. Taking a look at the number of vertices of $\mathrm{T}(S_n)$ and $\mathrm{T}(P_n)$, as $N = \sum \min\{\deg x, 2\}$, we have $n+1$ and $2(n-1)$, respectively. From these extremal values, we have:

$$\left\lceil \frac{N}{2} \right\rceil + 1 \leq n \leq N - 1. \tag{4}$$

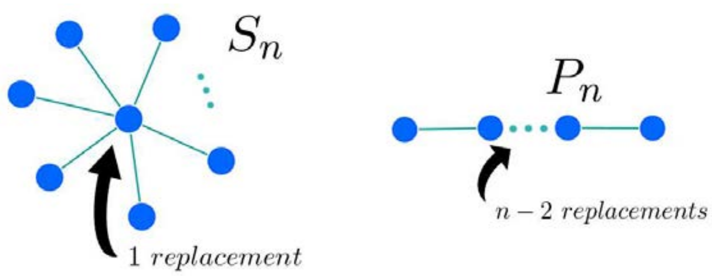


Figure 4: The extremal trees with respect to the number of vertices replaced by the transformation T are the star graph S_n and the path graph P_n.

Another way of obtaining some twin graphs from trees is by constructing the double graph of that tree. In words, the double graph $D(X)$ of a graph X is obtained by making a copy X' of X and connecting all neighbours of a vertex v in X to its correspondent vertex v' in X'. At the end of this process, every pair $v, v' \in D(X)$ will be a twin pair. If X is a tree of order $n \geq 2$, it is easy to see that $D(X)$ is a minimum 2-GC graph different from K_3 and Q_3, thus it is a twin graph.

In [Indulal and Gutman, 2008], [Ganie et al., 2014], and [Marino and Salvi, 2007], there are important properties of double graphs regarding their spectra. We state these properties below as lemmas.

Lemma 3. [Indulal and Gutman, 2008] Let $X \in \mathrm{G}_n$ be connected and denote by $a(\partial)$ the algebraic multiplicity of $\partial \in \Lambda(\mathbf{d}_X)$. The characteristic polynomial of the distance matrix of $D(X)$ is:

$$p(\mathbf{d}_{D(X)}, x) = (x+2)^n \prod_{\partial \in \Lambda(\mathbf{d}_X)} [x - (2\partial + 2)]^{a(\partial)}.$$

Lemma 4. [Ganie et al., 2014] Let $X \in \mathrm{G}_n$ be connected and denote by $a(\lambda)$ the algebraic multiplicity of $\lambda \in \Lambda(\mathbf{a}_X)$. The characteristic polynomial of the adjacency matrix of $D(X)$ is:

$$p(\mathbf{a}_{D(X)}, x) = x^n \prod_{\lambda \in \Lambda(\mathbf{a}_X)} (x - 2\lambda)^{a(\lambda)}.$$

Lemma 5. [Marino and Salvi, 2007] Let $X \in \mathrm{G}_n$ be connected and denote by $a(\mu)$ the algebraic multiplicity of $\mu \in \Lambda(\mathbf{l}_X)$. The characteristic polynomial of the Laplacian matrix of $D(X)$ is:

$$p(\mathbf{l}_{D(X)}, x) = \prod_{v \in \mathrm{V}(X)} (x - 2\deg v) \prod_{\mu \in \Lambda(\mathbf{l}_X)} (x - 2\mu)^{a(\mu)}.$$

Therefore, if a twin graph is a double graph of a tree, then its adjacency, distance and Laplacian spectra are fully characterized by the correspondent spectra of the tree. However, since not every twin graph is a double graph of a tree, the spectra of all twin graphs are not yet characterized. This justifies a computational study on the spectra of this class in order to obtain some data that can help us to understand how the spectra of twin graphs behave. In the next section we present such a computational study, without formal mathematical proofs. We have used the software Mathematica [Wolfram Research, 2016] for these studies.

3. Computational study of the spectra of twin graphs from 4 up to 14 vertices

The mapping T, defined in Proposition 1, and the operation of constructing double graphs are links between trees and twin graphs. The trees are the minimum-size connected (redundantly, 1-GC) graphs, whereas the twin graphs are the minimum-size 2-GC graphs. Both graph classes minimize the number of edges required for having the property of geodetic connectivity, but the twin graphs are of a higher connectivity.

This topological property can also be observed by spectral metrics of graphs. By analyzing the spectra of the adjacency, distance and Laplacian matrices, one can notice that some eigenvalues of these matrices clearly differentiate twin graphs from trees. In the following analysis we consider, as sample, all twin graphs and trees from 4 up to 14 vertices. The trees were generated by using the Nauty library [McKay, 1984], and the twin graphs were generated by the recursive process found in [Farley and Proskurowski, 1997] and stated in Definition 1. Table 1 presents the number of graphs of both classes, for each number of vertices in that range.

n	4	5	6	7	8	9	10	11	12	13	14
trees	2	3	6	11	23	47	106	235	551	1301	3159
twin graphs	1	1	2	2	4	9	13	13	23	35	63

Table 1: Cardinality of the sets $\mathcal{T}_n$ and $\mathbb{I}_n$ for $4 \leq n \leq 14$.

Normalized histograms for the eigenvalues of trees and twin graphs of order 14 are shown in Figures 5, 6, and 7 for the adjacency, distance and Laplacian matrices, respectively. They indicate that all the studied eigenvalues have different distributions for both trees and twin graphs. It means that, despite both graph classes being minimal, for their geodetic connectivities, with respect to the number of edges, the spectra reflects this increment in geodetic connectivity.

Figure 8 presents the adjacency index (λ_1) and the Laplacian eigenvalue known as algebraic connectivity (μ_{n-1}) for all trees and twin graphs from 4 up to 14 vertices, with respect to the

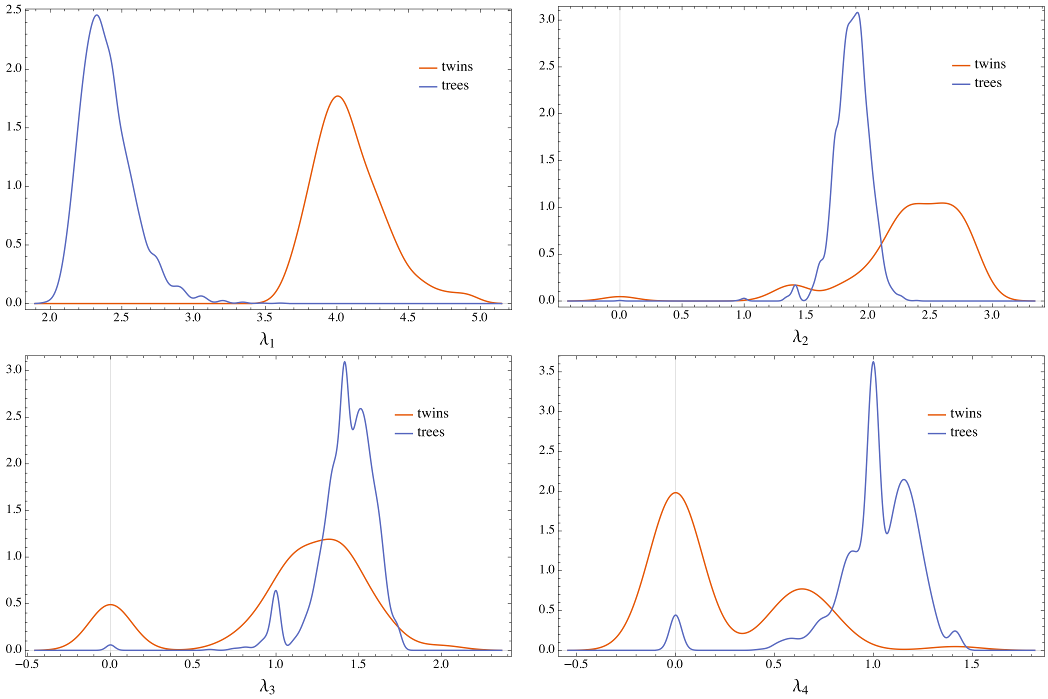


Figure 5: Normalized histograms for the first four eigenvalues of the adjacency matrix (λ_1, λ_2, λ_3, λ_4) for twin graphs and trees of 14 vertices. The separation between the indices (λ_1) of trees and twin graphs can be explained by Equation (5).

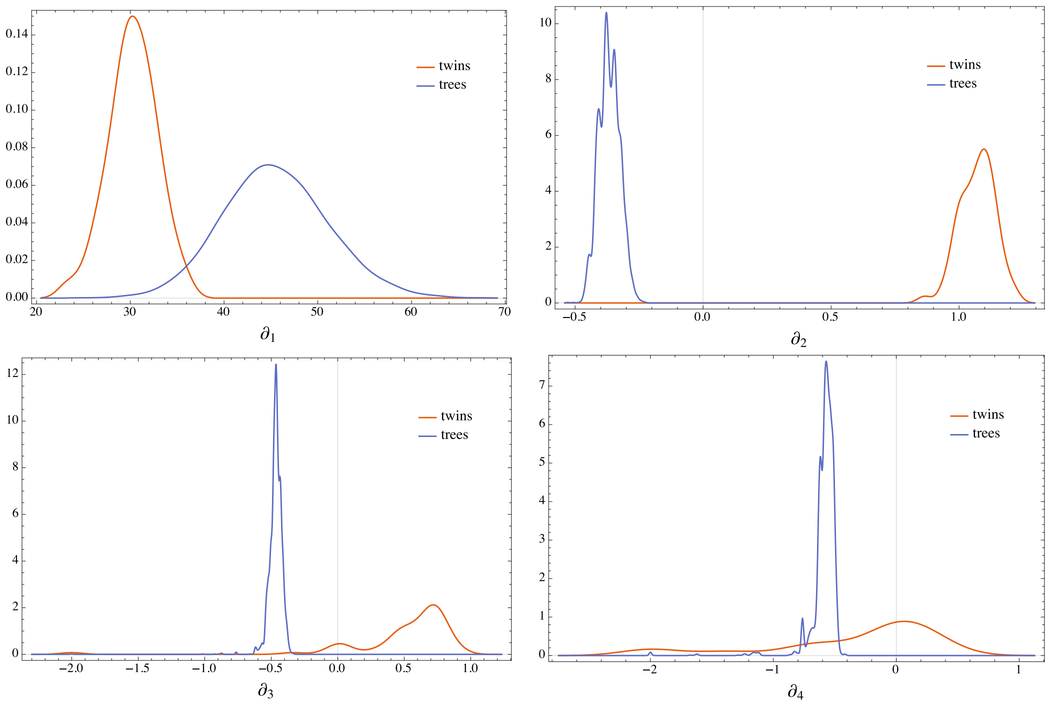


Figure 6: Normalized histograms for the first four eigenvalues of the distance matrix (∂_1, ∂_2, ∂_3, ∂_4) for twin graphs and trees of 14 vertices. Worth mentioning, there is a large gap between the second distance eigenvalue of trees and twin graphs. This can be used to identify whether a graph from the studied sample belongs to the class of trees or twin graphs.

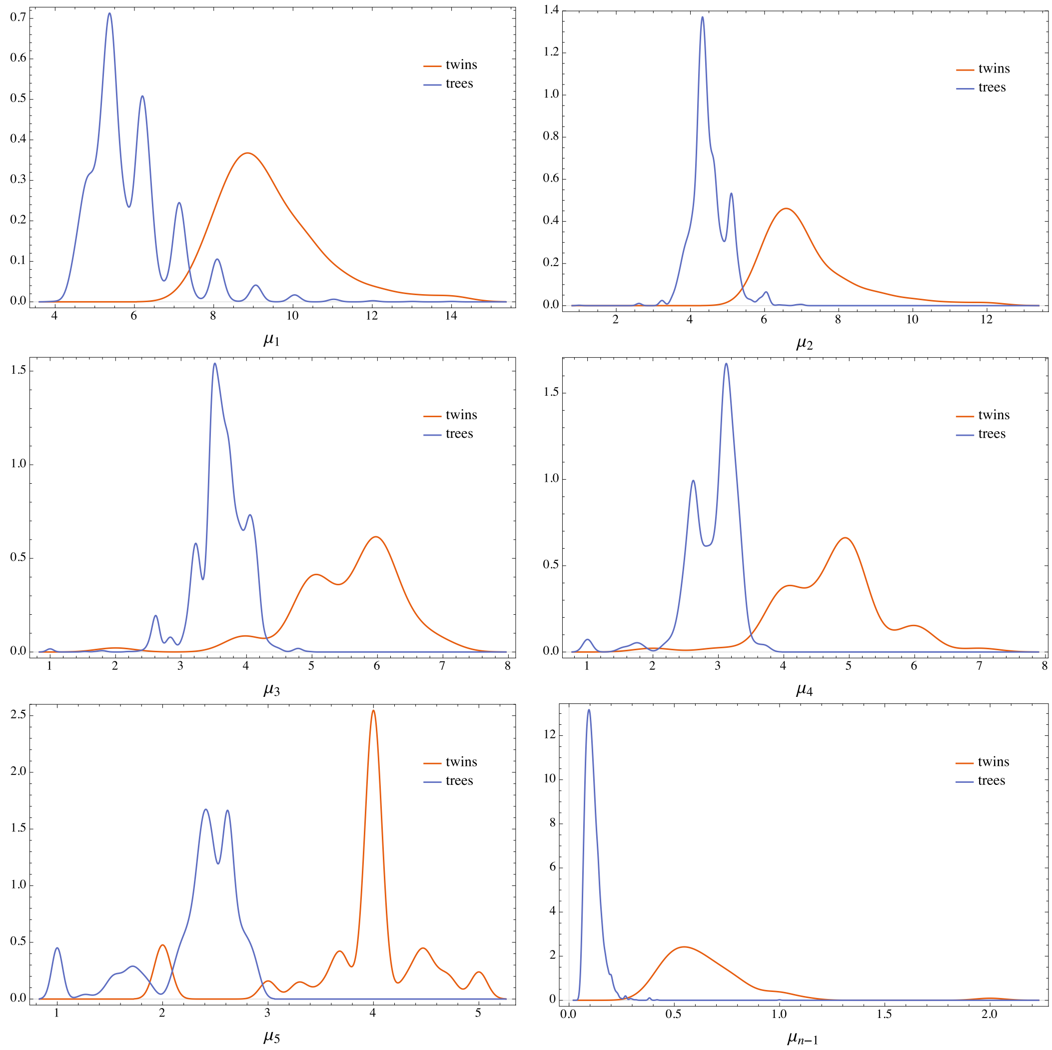


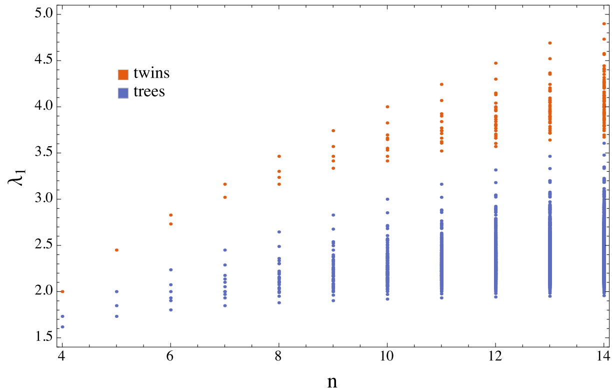
Figure 7: Normalized histograms for the first five eigenvalues of the Laplacian matrix (μ_1, μ_2, μ_3, μ_4, μ_5) for twin graphs and trees of 14 vertices, and the algebraic connectivity (μ_{n-1}).

number of vertices. Notice that λ_1 is greater for every twin graph when compared to the trees, as shown in Figure 8(a). It is due to the following upper bound [Abreu et al., 2007], which holds for every simple connected graph:

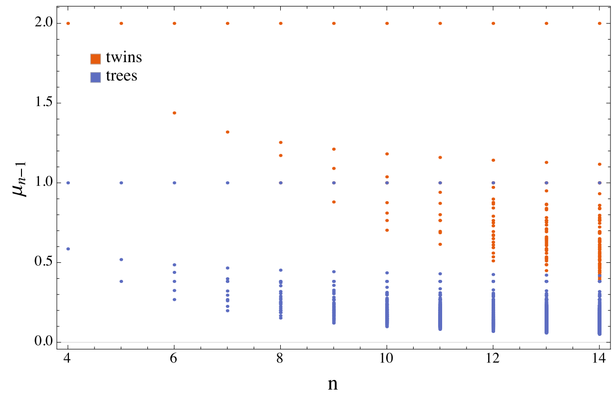
$$\lambda_1 \leq \sqrt{2m\left(1-\frac{1}{n}\right)}. \tag{5}$$

According to Figure 6, the first four distance eigenvalues exhibit a clear difference between the spectral behavior of the trees and twin graphs. In particular, for our sample, the signal of the second distance eigenvalue (∂_2) can be used to determine whether the graph is a tree or a twin graph. This eigenvalue, for both twin graphs and trees, is located in a narrow interval along the vertical axis, as emphasized in Figure 9.

The algebraic connectivity (μ_{n-1}) also shows a distinction when trees and twin graphs are compared. In general, twin graphs have greater algebraic connectivity than trees, as observed in Figure 7 and emphasized in Figure 8(b). It can be explained by the following result due to [Fiedler, 1973]: for a fixed number of nodes, as the number of edges grows, the algebraic connectivity does not decrease.



(a) Adjacency index. By using Equation (5) we can compare the adjacency index of trees and twin graphs of the same order. Since the order n is fixed, consider $k = \sqrt{2(1 - 1/n)}$. For a tree on n vertices, the number of edges is $n - 1$, and therefore $\lambda_1(\mathcal{T}_n) \leq k\sqrt{n-1}$. On the other hand, for a twin graph of the same order we have $\lambda_1(\mathbb{I}_n) \leq k\sqrt{2(n-2)}$.



(b) Second smallest Laplacian eigenvalue, the algebraic connectivity. It is known that for $n \geq 6$ all trees except the stars satisfy $\mu_{n-1} < 0.49$. Considering the same order n, both graph classes have well defined number of edges, which are $m(\mathcal{T}_n) = n - 1$ and $m(\mathbb{I}_n) = 2(n-2)$. We have $m(\mathbb{I}_n) - m(\mathcal{T}_n) = n - 3$. It is known that for a fixed order as the number of edges grows the algebraic connectivity does not decrease.

Figure 8: Adjacency index and algebraic connectivity for all trees and twin graphs from 4 up to 14 vertices, with respect to the number of vertices.

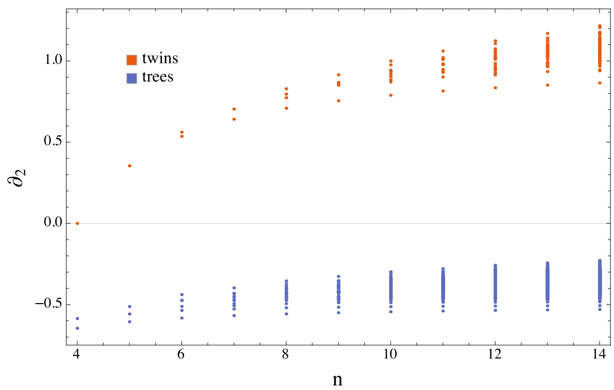
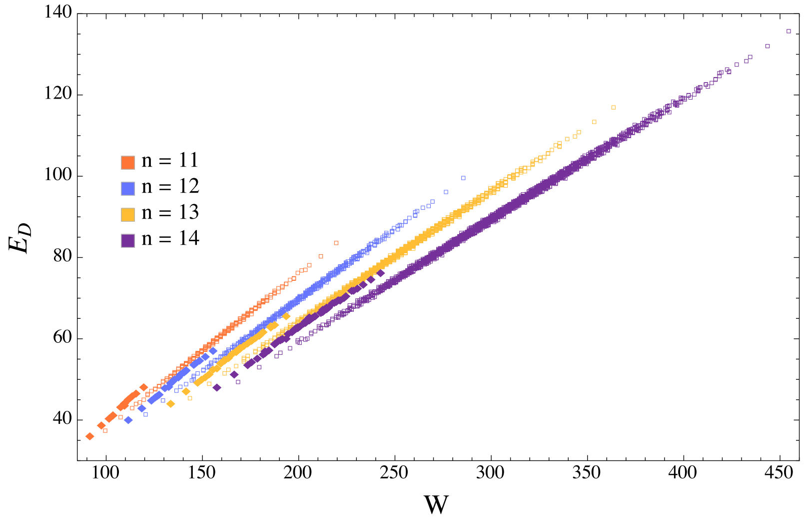
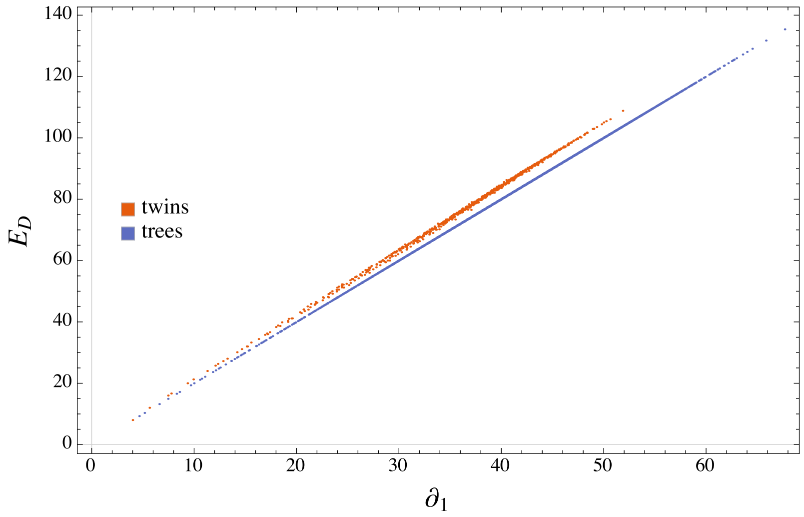


Figure 9: Second distance eigenvalue for all trees and twin graphs from 4 up to 14 vertices, with respect to the number of vertices.

The following analysis associates spectral and non-spectral metrics of the distance matrix with the first distance eigenvalue of trees and twin graphs. Figure 10(a) shows the distance energy E_D as a function of the Wiener index W, whereas Figure 10(b) shows E_D as a function of the distance index ∂_1.



(a) Distance energy as a function of the Wiener index for trees and twin graphs of 11 up to 14 vertices. The blank square points represent the trees whereas the filled ones represent the twins. We can observe that both trees and twin graphs behave almost linearly.



(b) Distance energy as a function of the first distance eigenvalue, the distance index, for all twin graphs and trees of 4 up to 14 vertices. Trees behave exactly linearly. In fact, the exact expression is $\mathrm{E}_D(\mathcal{T}_n) = 2\partial_1(\mathcal{T}_n)$. Also, it is very interesting that twin graphs behave almost linearly.

Figure 10: Linear relationships between the distance energy E_D, Wiener index W and the distance index ∂_1 of trees and twin graphs.

In Figure 10(a) a curious linear behavior is observed for both trees and twin graphs. By fitting the curves, Approximations (6) and (7) were obtained for the trees $X_{\mathcal{T}} \in \mathcal{T}_n$ and the twins $X_{\amalg} \in \amalg_n$, respectively, for $11 \leq n \leq 14$.

$$\mathrm{E}_D(X_{\mathcal{T}}) \approx K_{\mathcal{T}}(n)\,\mathrm{W}(X_{\mathcal{T}}); \tag{6}$$
$$\mathrm{E}_D(X_{\amalg}) \approx K_{\amalg}(n)\,\mathrm{W}(X_{\amalg}), \tag{7}$$

where $K_{\mathcal{T}}(n)$ and $K_{\amalg}(n)$ are constants depending only on the number of vertices of the graphs. Fitting the constants, we have $K_{\mathcal{T}}(n) = 4.38147/n$ and $K_{\amalg}(n) = 4.15213/n$.

However, in Figure 10(b), an (order-independent) linear relationship between E_D and ∂_1 is observed for both trees and twin graphs. That behaviour was already expected for trees, but it is curious for twin graphs. By fitting the curve for twin graphs, we have:

$$\mathrm{E}_D(X_{\amalg}) \approx 2.10756\,\partial_1(X_{\amalg}). \tag{8}$$

By Approximations from (6) to (8) we have for both trees and twin graphs:

$$\partial_1(X) \approx \frac{2.08}{n}\,\mathrm{W}(X). \tag{9}$$

By computing the value of $\partial_1(X)\,n/\,\mathrm{W}(X)$ for all connected graphs on 8 vertices and all trees and twin graphs up to 14 vertices, it was found that:

$$\frac{2\,\mathrm{W}(X)}{n} \leq \partial_1(X) \leq \frac{2.1187\,\mathrm{W}(X)}{n}. \tag{10}$$

The left inequallity can be found in [Aouchiche and Hansen, 2014] and it holds for any connected graph G of order $n \geq 2$. The right inequallity was obtained computationally. It is worth to notice that, for both trees and twins up to 14 vertices and for all connected graphs on 8 vertices, the value of $\partial_1(G)\,n/\,\mathrm{W}(G)$ remains close to 2. The last result in Inequation (10) is very interesting, for it shows that the distance index of every connected graph on 8 vertices and trees and twin graphs from 4 up to 14 vertices can be found in that range.

4. Conclusion and future research

Twin graphs are (almost all) the minimum-size graphs for which all graph distances remain unchanged after deleting a vertex. Thus, whereas trees can become disconnected when a single vertex is removed, twin graphs suffer the minimum impact in the same situation, while using less edges than any other graph in the class of the 2-geodetically connected graphs. Also, the fact that there are 2 disjoint shortest paths between any pair of non-adjacent vertices, with the minimum number of edges, can be of interested in flow and routing problems.

By a well defined function and by the construction of double graphs, a tree can be transformed into a twin graph that contains that tree as an induced subgraph, preserving the previous connections. The spectra of twin graphs that are double graphs are given as a function of those of the trees that were transformed, but the relationship between the proposed function and the spectra is unknown.

After a computational study, it became evident that the difference in the geodetic connectivity between trees and twins is reflected in their spectra and in spectral metrics, such as the distance energy. As future research, we aim to describe completely the spectra of twin graphs by the spectra of the trees that will be mapped into twin graphs, also with the generalization of this mapping to other graph classes with the spectral characterization.

The value of $\partial_1(G)\,n/\,\mathrm{W}(G)$ that remained approximately constant to all trees and twins considered will be studied and described in future works, since a good result was obtained by its analysis: for every connected graph on 8 vertices and for every tree and twin graph of order 4 up to 14 the following inequality holds: $2 \leq \partial_1(G)\,n/\,\mathrm{W}(G) \leq 2.1187$.

Another interesting question to be investigated is how different is the spectrum of a twin graph when compared to a non-twin graph of the same order and size.

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