Optimal Resampled Frontier, the Modal Optimum and Examples

Marcos Huber Mendes
Pontifícia Universidade Católica do Rio de Janeiro, DEE – Departamento de Engenharia Elétrica
Rua Marquês de São Vicente, 225, Edifício Cardeal Leme, sala 401,
Gávea – Rio de Janeiro – RJ CEP 22451-900
hubermendes@decisionsupport.com.br

Reinaldo Castro Souza
Pontifícia Universidade Católica do Rio de Janeiro, DEE – Departamento de Engenharia Elétrica
Rua Marquês de São Vicente, 225, Edifício Cardeal Leme, sala 401,
Gávea – Rio de Janeiro – RJ CEP 22451-900
reinaldo@ele.puc-rio.br

Marco Aurélio Sanfins
Universidade Federal Fluminense
Rua Professor Marcos Waldemar de Freitas Reis, s/n,Campus do Gragoatá - Bloco H,
5º andar, sala 501, Niterói – RJ CEP 24210-201
marcosanfins@automata.uff.br

ABSTRACT

This paper presents contributions to the development and generalization of the Markowitz`s Portfolio Optimization Model [Markowitz 1952, 1956, 1959, 1991]. First, we define a new measure of risk considering all cross interrelationships between returns in addition to deviations above and below a reference target. This measure allows us to select Portfolios with higher returns compared to Markowitz`s Portfolio and others that consider only those partial moment deviations. Second, we use simulations by resampling the Portfolio Assets in order to consider the optimal distribution frequency. This allows to evaluate uncertainties inherent to Portfolio selection and the Optimal Resampled Frontier which provide assess to the Modal Optimum, giving a probability measure to the occurrence of the selected optimal. To solve the optimization the new measure of risk use a convex nonlinear optimization model (NLP), with lower computational consumption than non-smooth optimization models (NSP) needed for risk measures that consider partial moments.

KEYWORDS: Optimal, Modal, Resampled.

Paper topics: SIM, OA, GF

1. Efficient Frontier Model
1.1. E-V Efficient Portfolio Model (Markowitz, 1952)

The mathematical formulation of the E-V Efficient Portfolio [Markowitz 1952, 1956, 1976, 1991] model considers three basic premises: i) uncovered sales are not allowed; ii) the sum of the fractions to invest equals the capital available for investment; and iii) the Assets are correlated but not perfectly correlated, implying that diversification can reduce but not eliminate risk.

Based on these assumptions, the E-V Efficient Portfolio [Markowitz 1952, 1956, 1976, 1991] model seeks to find the Efficient Portfolio that offers lowest risk for each level of pre-defined Minimum Acceptable Expected Returns (MAR). Risk is measured as the variance of the returns of the Portfolio of Assets. The set of Efficient Portfolios for different levels of returns constitutes the Efficient Frontier (EF), which limits the feasible Portfolios.

Each investor should make the selection of a Portfolio from criteria that depend on the nature of the investor, since the best Portfolio for each investor depends on his own tolerance for risk [Markowitz 1952, 1956, 1976, 1991], this is because Portfolios with highs expected returns are not necessarily those with the lowest uncertainty of return.

1.2. The Mean-separated Target Deviations (MSTD) model

The Mean-separated Target Deviations (MSTD) model [Kang et al. 1996], unlike the E-V Efficient Portfolios [Markowitz 1952, 1956, 1976, 1991] model, considers the risk as a joint measure of deviation below (BTD) and above (ATD) a certain target, by using the concept of non-central semi-moment or deviation around a target.

The MSTD risk measure is a generalization of measures that used the concept of Low Partial Moment (LPM) for risk, and the return as a function of Upper Partial Moment (UPM) [Holthausen 1981]. In particular, the MSTD model can be reduced to the semi-variance above and below a target, the risk measure proposed as in [Fishburn 1977], to the risk measure as in [Markowitz 1952, 1956, 1976, 1991], and to the risk measure as in [Bawa 1975].

However, the semi-moment of order 2 for a target does not assume a quadratic form, which prevents construction of the Portfolio semi-moment from the n semi-moments and covariate semi-moments of order 2. In general, for different orders degrees, one does not have a literal form that allows simplifying the computational complexity of a discontinuous and nonlinear algorithm for solving semi-moment models. This means that we have to use empirical data to solve the optimization model algorithm, without a literal form expression, which makes the optimization algorithm a non-smooth optimization model (NSP) with a complex solution and high computational consumption and which only provides a single viable solution or a local optimum.

1.3. Statistic Efficient Risk Measure

Define M as the matrix of the integral of squared deviations and cross deviations of Portfolio returns to a target t:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mathbf{x_i}-\mathbf{t})(\mathbf{x_j}-\mathbf{t})\mathbf{f_{ij}}(\mathbf{x_i},\mathbf{x_j})\mathbf{dx_i}\mathbf{dx_j}, \qquad (1)$$

where t is the target, or MAR, and fij represents the joint density probability function of the Assets i and j. Assuming i = j in (1), we obtain:

$$\boldsymbol{M(i,i)} = \int_{-\infty}^{\infty}(\boldsymbol{x_i}-\boldsymbol{t})^2\boldsymbol{f_i}(\boldsymbol{x_i})\boldsymbol{dx_i}$$

Proposition 1: The matrix M can be decomposed into the sum of $2n^2$ -2n +2 matrices, denoted by M^+, M^- and M_{ij}^{++}, M_{ij}^{+-}, M_{ij}^{-+}, M_{ij}^{--},with $1 \le i \le j \le n$, based on the returns of Assets and on the target t, such that:

- $M^+(i,i) = \int_t^{\infty}(x_i - t)^2 f_i(x_i)dx_i$, and $M^+(i,j) = 0, i \neq j$;
- $M^-(i,i) = \int_{-\infty}^{t}(xi - t)^2 f_i(x_i)dx_i$, and $M^-(i,j) = 0, i \neq j$;
- $M_{ij}^{++}(i,j) = M_{ij}^{++}(j,i) = \int_t^{\infty}\int_t^{\infty}(x_i - t)(x_j - t)f_{ij}(x_i, x_j)\,dx_i dx_j$, and 0 for the others entries;
- $M_{ij}^{+-}(i,j) = M_{ij}^{+-}(j,i) = \int_{x_j=-\infty}^{t}\int_{x_i=t}^{\infty}(x_i - t)(x_j - t)f_{ij}(x_i, x_j)dx_i dx_j$, and 0 for the others entries;
- $M_{ij}^{-+}(i,j) = M_{ij}^{-+}(j,i) = \int_{x_j=t}^{\infty}\int_{x_1=-\infty}^{t}(x_i - t)(x_j - t)f_{ij}(x_i, x_j)dx_i dx_j$, and 0 for the others entries;
- $M_{ij}^{--}(i,j) = M_{ij}^{--}(j,i) = \int_{x_j=-\infty}^{t}\int_{x_i=-\infty}^{t}(x_i - t)(x_j - t)f_{ij}(x_i, x_j)dx_i dx_j$, and 0 for the others entries.

The properties of matrices and the split of integrals easily perform the deduction of this result. The empirical formulas analogous to these matrices are respectively:

- $M_{ii}^{+}(i,i) = M_{ii}^{+}(i,i) = \sum_{\substack{1 \le s \le z, \\ x_{is} > t,}} (x_i s - t)^2$, and 0 for the others entries;
- $M_{ii}^{-}(i,i) = M_{ii}^{-}(i,i) = \sum_{\substack{1 \le s \le z, \\ x_{is} < t}} (x_i s - t)^2$, and 0 for the others entries;
- $M_{ij}^{++}(i,j) = M_{ij}^{++}(j,i) = \sum_{\substack{1 \le s \le z, \\ x_{is} > t, \\ x_{js} > t}} (x_i s - t)(x_{js} - t)$, and 0 for the others entries;
- $M_{ij}^{+-}(i,j) = M_{ij}^{+-}(j,i) = \sum_{\substack{1 \le s \le z, \\ x_{is} > t \\ x_{js} < t}} (x_{is} - t)(x_{js} - t)$, and 0 for the others entries;
- $M_{ij}^{-+}(i,j) = M_{ij}^{-+}(j,i) = \sum_{\substack{1 \le s \le z, \\ x_{is} < t, \\ x_{js} > t}} (x_i s - t)(x_{js} - t)$, and 0 for the others entries;
- $M_{ij}^{--}(i,j) = M_{ij}^{--}(j,i) = \sum_{\substack{1 \le s \le z, \\ x_{is} < t \\ x_{js} < t}} (x_{is} - t)(x_{js} - t)$, and 0 for the others entries.

We construct the proposed risk measure, referred to as Statistic Efficient (SE) risk measure, based on the matrix M. Since M, in addition to taking into account the deviations above and below the target t, considers all interrelationships between these deviations, SE also allows the solution of the optimization model through a literal form based on the returns of the Assets and the target t.

Proposition 2: The squared deviation from the return $\mathbf{r_{ps}}$ on a Portfolio of Assets A_1, A_2, ..., A_n in relation to a minimum return $\mathbf{t}$ can be written as a function of squared deviations and cross squared deviations of the returns $x_i$of each Asset A_i

Proof: Indeed,

$$\sum_{s=1}^{z}\{(r_{ps} - t)^{\,2}\} = \sum_{s=1}^{z}\{(\sum_{i=1}^{n} w_i(x_{is} - t))^2\}$$

$$= \sum_{s=1}^{z}\{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j (x_{is} - t)(x_{js} - t)\}$$
$$= \sum_{s=1}^{z}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \{(x_{is} - t)(x_{js} - t)\}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j m(i,j)$$

Definition: The measure of Statistic Efficient risk, denoted by SE, can be defined as:

$$\textbf{SE} = \lambda M^{-} + \gamma M^{+} + \sum_{1\le i<j\le n}(\alpha_{ij}M_{ij}^{++} + \beta_{ij}M_{ij}^{+-} + \delta_{ij}M_{ij}^{-+} + \eta_{ij}M_{ij}^{--}) \qquad (2)$$

We can extend the sum of squared deviations and cross-squared deviations of returns from the target t to any order degree without loss of the properties displayed. One of the advantages of SE is that it provides a high dimensional space in the search region formed by the optimization problem and solved by using a convex nonlinear algorithm (NLP). In the following sections, we also use simulations to estimate the SE risk measure by using a convex nonlinear optimization algorithm (NLP) that, even though containing a higher number of parameters, requires much less computational consumption compared to non-smooth optimization algorithm (NSP) models.

The E-V Efficient Portfolios [Markowitz 1952, 1956, 1976, 1991] model and its risk metrics evolution as MSTD [Kang et al. 1996] model, by not taking into account the frequency distribution of the optimization results, do not allow calculating how frequently the optimal Portfolio occurs, so it is not possible to know how often the optimal Portfolio will be (was) optimal in future (past) periods. Among other reasons, due to the uncertainties in the estimates of position and dispersion measures of Assets, the Portfolio calculated by the E-V Efficient Portfolios [Markowitz 1952, 1956, 1976, 1991] model fails to attain the optimal parameters [Jobson and Korkie 1976]. In addition, because Portfolio optimization models are based on the mean, variance and correlation between Asset returns, persons often mistakenly assume normal distributions for the respective returns. The assumption of normality, however, is not a convenient assumption, the distributions of Assets returns are generally asymmetric and leptokurtic [Kang et al. 1996].

We did not find in the literature a Portfolio selection model that considers the frequency of occurrences as the model starting point and that considers as result the analysis of the frequency distribution of the optimal results, instead of statistical position and deviation measures. We note here that the Resampled Efficient Frontiers [Michaud et al. 1998, 2003, 2013] model result is an average of resampled E-V Efficient Portfolios [Markowitz 1952, 1956, 1976, 1991] models, in other words, a statistical position measure. We can show through simulations that the consideration of the distribution frequencies of Ex-Ante and Ex-Post evaluation of the optimal parameters can provide the frequency of occurrence of the optimal Portfolio.

2. Design of Simulations

We consider the Modal Optimum from the frequency of occurrence of all the optimums selected for a group from a cluster analysis. In this case, we calculate the frequency of occurrence of an optimum by means of simulation where for each iteration we carried out an optimization. We consider the window of observations of Assets to be a unit window, where the expected return of the Portfolio is given by the return at each iteration. Thus, it is possible at each iteration to evaluate the optimal parameters, such as the percentage of capital invested in each Asset. We call the Portfolio return calculation with window for one period a naive Portfolio, since implicitly we assume that the current return is the best forecast for the average returns of the Assets. This hypothesis is similar to the assumption made for building the U-Theil statistic by [Fildes 1992], exhibited initially by [Theil 1967].

By performing the simulation, which considers all possible combinations of Assets and calculates from each iteration all possible optimal values, we obtain the frequency distributions of the optimal parameters. From the method of clusters, we define the centroid of the group with highest frequency of optimal results as the representative of the Modal Optimum location and the frequency of optimums in that group as the Modal Optimum frequency. Thus, the percentage invested in each Asset, the Asset return and the Asset risk are evaluated from the centroid of the group with highest frequency, which we defined as the Modal Optimum. We define the method to obtain the Modal Optimum as the Distribution Efficient Method (DEM). Thus, we obtain the Portfolio defined as Statistic and Distribution Efficient (SDE) from applying the Statistic Efficient (SE) risk measure on the Distribution Efficient Method, which defines the Modal Optimum.

We define the calculated optimal number of groups used in the cluster method by the V-Fold Cross Validation method [Burman 1989]. In the process of exhibiting the results obtained by the SDE Portfolios, we use the optimal investment fraction defined by the DEM together with the holdout method. We use a data window for training, another data window for testing, and both for analyzing the performance of the SDE Portfolios. Specifically, we measure the SDE Portfolio performance, generated from the simulation, optimization and cluster analysis, by the Ex-Post and Ex-Ante analysis, applied to the data used to generate the Portfolio and to the holdout data, respectively. We will compare the results obtained by the SDE Portfolio model to the results obtained from the E-V Efficient Portfolio [Markowitz 1952, 1956, 1976, 1991] model and from the MSTD Portfolio [Kang et al. 1996] model, with all them using the DEM to obtain the Modal Optimum.

3. Results

3.1. Metrics for Portfolio Efficiency

For a wider comparison among the SE Portfolio, the SDE Portfolio, the E-V Efficient Portfolio [Markowitz 1952, 1956, 1976, 1991] and the MSTD Portfolio [Kang et al. 1996], we introduce here additional criteria for Portfolio Efficiency in supplement to Markowitz's [Markowitz 1952, 1956, 1976, 1991] efficiency criterion definition:

Computational Efficiency – The more complex the model is, the more slowly the computational algorithm solves the model. Fast algorithms or models that are solved for less complex algorithms are more computationally efficient.

Solution Efficiency – A model that presents a global optimal solution is more efficient than a model that presents a local optimal solution, which in turn is more efficient than a model that presents only a feasible solution. In general, Computational and Solution Efficiency should come together.

Diversification Efficiency – Diversification can reduce but not eliminate the risk, and Portfolios with high expected returns are not necessarily those with the lowest uncertainty of return. Some models lose diversification as a way to get higher returns. To balance the comparison of different models, we will define an Index of Diversification (ID).

ID = $1 - \prod w_i$, where $\boldsymbol{w_i}$ is the fraction of capital invested in Asset $\boldsymbol{A_i}$ and for the coherence of the index, when $\boldsymbol{w_i} < 0.05$ that $\boldsymbol{w_i}$ is not considered in the formula. Thus, ID can always be calculated. Higher the ID better the Diversification Efficiency.

Standard Risk Efficiency – Different Portfolio models calculate different risk metrics and we need to standardize the metrics by dividing the risk for a particular MAR in the Efficienty Frontier of a Portfolio by the higher risk achieved by that Portfolio, so as to have a standard dimensionless metric to compare models. Usually this higher risk occurs when, in the Efficient Frontier, we select a return high enough so that no diversification in the Portfolio occurs. Higher the standardized Risk worst the Standard Efficiency.

Return Efficiency – We can compare the return of a different Portfolio models in different ways. Here we compare the return by:

- The highest return among models applying the Modal Optimum in a single period;
- The highest cumulative return, applying the Modal Optimum from each model, calculated from a certain past period, to the time evolution of the Assets previously considered for holdout in the Ex-Ante analysis of the optimization models;

- The best frequency distribution of return in the Ex-Post analysis;
- The best return for a specified MAR in his related Modal Optimum.

For an investment for a single period or long period the investor uses the optimal Portfolio, which is the Portfolio of Assets acquired for the next period and reinvested in each new period [Markowitz 1976].

Modal Efficiency or efficiency in frequency results. – According to the modal efficiency criterion, a model is more efficient if it presents, for the same desired MAR, an optimum with higher frequency of occurrence, the Modal Optimum. Therefore, we have that the efficiency criterion is determined by the frequency of occurrence of the Modal optimum. For the same desired MAR, the Portfolio method assigned to a cluster that has the highest frequency of optimums will be the Portfolio method with greatest Modal Efficiency.

3.2. Ex-Post comparison of Portfolios with three Assets of the São Paulo Stock Exchange

We first compare the SE Portfolio and the SDE Portfolio performance to the performance obtained from the E-V Efficient Portfolio [Markowitz 1952, 1956, 1976, 1991] and the MSTD Portfolio [Kang et al. 1996] models taking into account three Assets of the São Paulo Stock Exchange in Brazil (Bovespa), named **PETR4**, **BBDC4** and **GGBR4**. This analysis considers a history Index window of 100 closing price (returns) of August 1, 2005 (100%) until December 22, 2005.

We compared the Portfolios models considering the Efficient Frontier of the E-V Efficient [Markowitz 1952, 1956, 1976, 1991] model and setting the MAR by using the Security Market Line (SML) as independently developed on the Capital Asset Pricing Model (CAPM), by [Treynor 1961, 1962], [Sharpe 1964], [John Lintner 1965a,b] and [Mossin 1966]. We consider a MAR equals 128% for models comparison, in a Efficient Frontier varying from to 100.00% to 158.20%, that means a risk free Asset of 102.50% in the period of the window of 100 dayly returns, which is equivalent to a risk free Asset with a return of 109.00% per year. During the 100 dayly returns the Selic tax (average adjusted rate of daily financing determined in the Special System of Settlement and Custody for federal securities by the Central Bank of Brazil) had an annual average of 119.14%.

The three Assets statistics are presented in Table 1 below for the Assets Index and for the Assets Index made stationary by aplying finite difference of order one:

Table 1 – Three Assets of the São Paulo Stock Exchange

Descriptive Statistics %					
Index					
	Mean	Variance	Std Deviation	Minimum	Maximum
PETRA4	118.30	76.43	8.74	100.00	133.09
BBDC4	134.99	452.47	21.27	100.00	182.27
GGBR4	127.90	175.77	13.26	100.00	158.20
Index Made Stationary					
	Mean	Variance	Std Deviation	Minimum	Maximum
PETRA4	0.33	5.93	2.44	-6.65	6.07
BBDC4	0.72	8.98	3.00	-7.61	7.19
GGBR4	0.59	7.59	2.75	-5.89	8.27

Spearman Correlation			
Index			
	PETRA4	BBDC4	GGBR4
PETRA4	1.00	0.66	0.92
BBDC4	0.66	1.00	0.62
GGBR4	0.92	0.62	1.00
Index Made Stationary			
	PETRA4	BBDC4	GGBR4
PETRA4	1.00	0.38	0.41
BBDC4	0.38	1.00	0.45
GGBR4	0.41	0.45	1.00

The three Assets are among the most traded stocks of the Bovespa, usually preferred by the managers. Two of them have low risks (standard deviation) in comparison with other traded Assets and one has high risks. Table 2, Panels A, B, C and D below presents a performance comparison of the SE model, the E-V Markowitz [1952, 1956, 1976, 1991] model and the MSTD [Kang et al. 1996] model by the defined efficiency criterions, were in the field Order “1” is the best and “3” is the worst performance.

We will see that, in general, SE and SDE Portfolios outperforms the related models, if we consider Computacional, Return, ID, Standard Risk and Modal Efficiency criterions together. As we see below the MSTD [Kang et al. 1996] model have the worst Computacional Efficiency and less Return, ID and Standard Risk Efficiency than the SE Model.

Moreover, below we analyze the Ex-Post models frequency distribution using the Modal Optimum by aplying the DEM method to the models. The distribution of the three Bovespa Assets are obtained making the series stationary, with the finite difference of order one, and calculating the histogram of each series, since these do not fit with goodness to any theoretical distribution usually fited for Asset distribution, with asymmetric and leptokurtic returns [Kang et al. [1996]. The simulation of the three Bovespa Assets is performed by simulating the histograms of each stationary Asset. For models comparison we set the MAR equals 4,0%, by the SML in a Efficient Frontier varying from to 0.56% to 6.0%, that means a risk free Asset of 2.50% in the period of the window of 100 dayly stacionary returns.

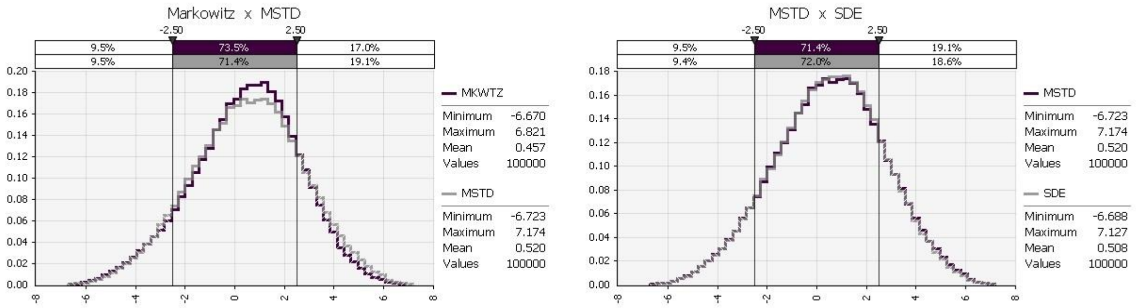
Table 2, Panels A, B, C and D

Panel A: Computacional Efficiency	Method	Seconds		Order
E-V Markowitz	QP	0.11		1
MSTD	NSP	120.40		3
SE	NLP	0.88		2
Panel B: Diversification Efficiency		**ID**		**Order**
E-V Markowitz		0.04%		3
MSTD		75%		2
SE		97%		1
Panel C: Return Efficiency- SinglePeriod		**Return**		**Order**
E-V Markowitz		128		2
MSTD		128		2
SE		129		1
Panel D: Standard Risk Efficiency	**Risk**	**Maximum Risk**	**Standard Risk**	**Order**
E-V Markowitz	13.44	21.27	63%	2
MSTD	9.86	14.14	70%	3
SE	55.13	141.9	39%	1

In the Ex-Post analysis, we applied the Modal Optimum fraction to invest in each Asset, determined by the compared Portfolio models, to measure the Portfolios returns generated by a simulation of 100,000 iterations, which allows the calculation of simulated variables with accuracy of 1% with 99% confidence. We re-sampled the historic window of 100 closing returns prices of the correlated histograms for the three Bovespa Assets.

Always considering the histograms and the correlation of the three Ibovespa series made stationary, with the finite difference of order one, the probability of being greater than 2.5% is 17.0% for the E-V Markowitz [Markowitz 1952, 1956, 1976, 1991] model, 19.1% for the MSTD [Kang et al. 1996] model and 18.6% for the SDE model. The E-V Efficienty [Markowitz 1952, 1956, 1976, 1991] model presents a maximum return of 6.82%, the MSTD [Kang et al. 1996] model presents a maximum return of 7.17%, and the SDE model presents a maximum return of 7.12%. These comparisons follow in Figure 1.

Figure 1 – Ex-Post Comparison of Markowitz, MSTD. and SDE models



To calculate the Modal Optimum and it frequency we used points on the efficient frontier with a MAR ranging from 0.56% to 6.00%. For being more detailed and comprehensive regarding the results, we show in Table 3 below the return, the risk and the frequency values of the Modal Optimum for each considered models and MAR values.

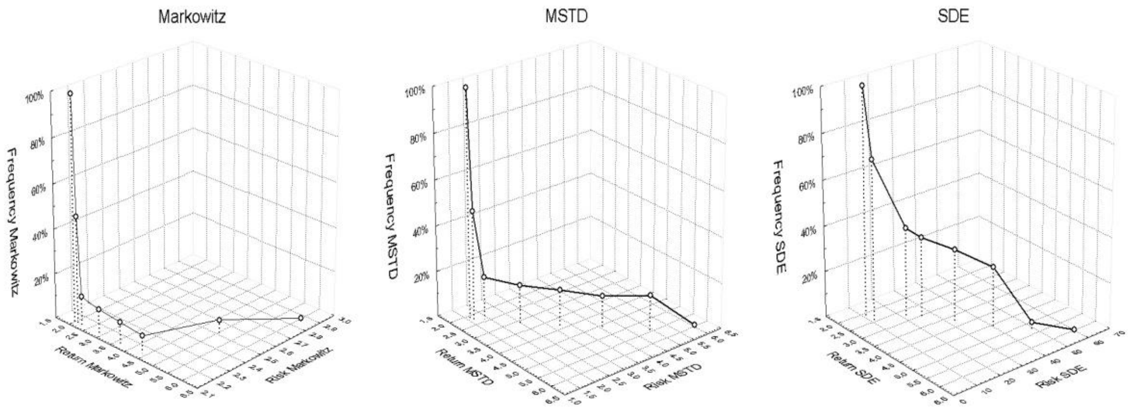
We can see that, for a specified MAR, the returns and the frequencies of the Modal Optimum of the SDE model most ofen outperforms the results for the Markowitz [Markowitz 1952, 1956, 1976, 1991] model and for the MSTD [Kang et al. 1996] model.

Table 3 – Modal Optimum Frequency

Efficienty Frontier MAR and Portfolio Models Returns, Risks and Frequency									
MAR	Return Markowitz	Risk Markowitz	Frequency Markowitz	Return MSTD	Risk MSTD	Frequency MSTD	Return SDE	Risk SDE	Frequency SDE
0.56	1.97	2.14	100%	2.09	1.63	100%	2.10	11.30	100%
0.68	2.13	2.14	37%	2.21	1.69	39%	2.45	11.66	59%
0.72	2.52	2.14	26%	2.75	1.71	26%	2.53	11.78	40%
1.00	2.23	2.14	12%	2.40	1.87	19%	2.75	22.08	39%
2.00	2.86	2.14	10%	3.10	2.51	17%	3.01	26.03	35%
3.00	3.62	2.14	9%	3.84	3.25	16%	3.74	32.63	32%
4.00	4.39	2.15	8%	4.56	4.08	14%	4.54	40.63	26%
5.00	5.00	2.52	6%	5.45	4.95	15%	5.28	49.33	3%
6.00	6.00	2.87	0%	6.00	6.03	0%	6.00	59.95	0%

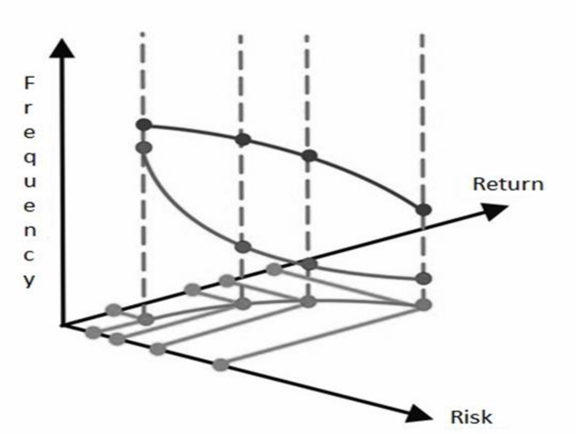
Figures 2 below present the results of Table 3 above, for each model, in three-dimensional graphs. The graphs show the Efficient Frontier, risk versus return, with the frequency as a third dimension. The third dimension is the frequency of optimal returns associated to the centroid of the cluster with highest frequency, for the MAR ranging from 0.56% to 6.00%. As seen before, using cluster analysis, we relate the Modal Optimum to the cluster with highest frequency. The figure below shows the Ex-Post analysis of the three models analyzed for frequency of occurrence of the Modal Optimum,

Figure 2 – Markowitz, MSTD and SDE Frequency for Modal Optimum of Models



Regarding the frequency of occurrence of the Modal Optimum, with the MAR ranging from 0.56% to 6.00%, we note that the E-V Efficienty [Markowitz 1952, 1956, 1976, 1991] and the MSTD [Kang et al. 1996] models show almost exponential decay with increasing restriction conditions of the MAR. The SDE model features a decay near the geometric one. Figure 3 below presents these patterns.

Figure 3 - Decay of Modal Optimum Frequency with Increasing MAR



3.3. Ex-Ante comparison of Portfolios with usual procedures for Portfolio Assets of the São Paulo Stock Exchange

We compare the SDE Portfolio performance to the returns obtained from the E-V Efficient Portfolio [Markowitz 1952, 1956, 1976, 1991] and the MSTD Portfolio [Kang et al. 1996] models considering a Portfolio with Assets of the Bovespa, select by usual investor procedures.

This analysis considers a history window of 100 closing price (returns) of August 1, 2005 until December 22, 2005 to calculate the Modal Optimum Portfolios and another sequential window of 395 returns, for holdouts, to be used on the Ex-Ante analysis, of December 23, 2005 until July 31, 2007.

The Portfolio selection chose Assets from the Ibovespa (a theoretical Portfolio of Assets prepared in accordance with the criteria set out in a method of the Bovespa). The objective of the Ibovespa is to be the indicator of the average performance of the prices of most traded and representative Assets in the Brazilian stock market. The final Ex-Ante results also present a comparison with the Ibovespa Portfolio. The selected Assets are among the most traded Assets of the Bovespa, usually preferred by managers. We chose Assets with high liquidity and high weight in the theoretical Portfolio of the Ibovespa. We also chose Assets that will account for nearly 50% of the Ibovespa.

This analisys considers two windows, one with 100 returns for modeling the Portfolio, and another with 395 holdouts used in the Ex-Ante analysis. The range of the study comprises a high volatility (risk) market with two stress moments. The select Assets are PETR4, VALE5, BBDC4, USIM5, ITAU4, GGBR4, TNLP4, CSNA3, UBBR11, ITSA4, CMIG4 and BRKM5. After making the series stationary, during the first window of 100 returns used for the Portfolios models calculus, we classified the selected Assets by their annual standard deviation as low volatility and high volatility Assets, as shown by the standard deviation of Assets, presented in Table 4 below.

Table 4 – Assets' Volatility

Standard Deviation	Low Standard Deviation							High Standard Deviation				Not Used
	BRKM5	CMIG4	VALE5	ITAU4	PETR4	ITSA4	CSNA3	GGBR4	UBBR11	BBDC4	USIM5	TNLP4
	4.79	4.94	5.00	5.77	5.93	6.00	6.06	7.59	8.47	8.98	9.62	2.14

In addition to the usual restrictions to the Markowitz Portfolio model, we also considered: i) not using the Asset TNLP4, which presents problems with minority shareholders; ii) that the percentage of an Asset with low volatility should always be less than 35%; and iii) that the percentage of an Asset with high volatility should always be less than 25%.

For evaluate the Modal Optimum we generated a simulation of 100,000 iterations, which allows the calculation of simulated variables with accuracy of 1% with 99% confidence. We resampled the historic distribution of the window of 100 closing prices of the correlated Asset returns for the select Assets from the Bovespa. The distribution of the eleven Bovespa Assets is obtained making the series stationary and calculating the histogram of each series, the simulation of the eleven Bovespa Assets is performed by simulating the histograms of each Asset. For models comparison we set the MAR equals 3,0%, by the SML in a Efficient Frontier varying from to 0.33% to 6.0%, that means a risk free Asset of 2.50% in the period of the window of 100 dayly stacionary returns.

In the Ex-Ante analysis we applied the Modal Optimum fraction to invest in each Asset, determined by the compared Portfolio models and evaluated using the first window of 100 returns, to the second window of 395 holdouts returns. The investor uses the Modal Optimum Portfolio obtained by the DEM method, which will be the Portfolio of Assets acquired for the next period and reinvested in each new period.

Making the beginning of the holdout period equals 100%, Figure 4 below shows the results for applying the Modal Optimum of each model to the second window of 395 holdouts returns, the graph also include the Ibovespa holdout returns.

Figure 4 – Ex-Ante Analysis

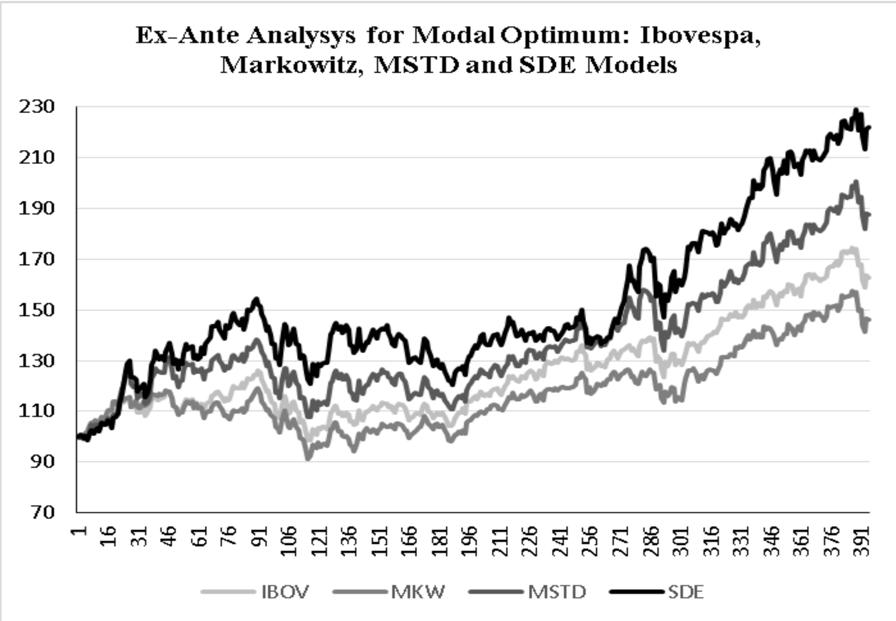


Table 5 below shows the index Return results for the E-V Efficient Portfolio [Markowitz 1952, 1956, 1976, 1991], the MSTD Portfolio [Kang et al. 1996], the SDE models and for the Ibovespa at the end of the 395 holdout window, making the beginning of the holdout period equals 100%.

Table 5 – Results at the End of the Holdout Window

Ibovespa	Markowitz	MSTD	SDE
162.56%	145.91%	187.34%	221.79%

4. Conclusion

The evolution of the E-V Efficient Portfolio [Markowitz 1952, 1956, 1976, 1991] for models based on risk measures that use partial moments, downside risk, upside risk [Bawa 1975] and [Fishburn 1977], LPM [Holthausen 1981], BTD and ATD [Kang et al. 1996] does not provide a solution to the optimization problem with a literal expression, but transforms the optimization algorithms into a non-smooth optimization model (NSP). These kinds of models have a complex solution with high computational consumption and provide only a single viable solution or a local optimum.

We presented a new measure of risk that considers all cross interrelationships between returns in addition to deviations above and below a reference target. Our measure provides a high dimensional space in the search region formed by the optimization problem and solved by the optimization algorithm. This allows us to obtain solutions with higher returns compared to the E-V Efficient Portfolio [Markowitz 1952, 1956, 1976, 1991] model and models as MSTD [Kang et al. 1996] that consider only ATD and BTD deviations. We also have as solution of the optimization problem a convex and nonlinear optimization model (NLP) that presents lower computational consumption compared to non-smooth optimization model (NSP).

Moreover, we used simulations that add to the usual optimization procedures by resampling the Portfolio in order to consider the returns distributions in addition to the position and dispersion measurements commonly used. This procedure allowed us to evaluate uncertainties inherent to the process of Portfolio selection and access the Modal Optimum, giving a probability measure to the occurrence of the selected optimal. In general, our paper makes contributions to the development and generalization of the Portfolio Optimization Models and sheds some light on the use of Modal Optimums in optimization analyses.

5. References

Bawa, V. S. (1975}. Optimal rules for ordering uncertain prospects. *Journal of Financial Economics*; Volume 2, Issue 1, March 1975, Pages 95–121.

Burman P. (1989). A Comparative Study of Ordinary Cross-Validation, v-Fold Cross-Validation and the Repeated Learning-Testing Methods. *Biometrika* Vol. 76, No. 3 (Sep., 1989), pp. 503-514.

Fildes, R. (1992). The evaluation of extrapolative forecasting methods. International *Journal of Forecasting*, 8, 81-98.

Glover et al. (1999). New advances for wedding optimization and simulation. *Simulation Conference Proceedings*, 1999 Winter, pp.255,260 vol.1, 1999.

Glover et al. (2001). Panel: simulation optimization: future of simulation optimization. WSC '01 Proceedings of the *33nd Conference on Winter Simulation*, Pages 1466-1469, IEEE Computer Society Washington, DC, USA ©2001.

Glover et al. (2003). Practical introduction to simulation optimization. *Simulation Conference,* 2003, Proceedings of the 2003 Winter, vol.1, no., pp.71,78 Vol.1, 7-10 Dec. 2003.

Glover et al. (2004). A unified modeling and solution framework for combinatorial optimization problems. *OR Spectrum*; March 2004, Volume 26, Issue 2, pp 237-250.

Holthausen, D. M, (1981). A Risk-Return Model with Risk and Return Measured as Deviations from a Target Return. *American Economic Review*, American Economic Association, vol. 71(1), pages 182-88, March.

JJobson, J.D.; and Korkie, B. (1981). Putting Markowitz Theory to Work. *The Journal of Portfolio Management*, Summer 1981, Vol. 7, No. 4: pp. 70-74.

Kang et al. (1996). A new efficiency criterion: The mean-separated target deviations risk model. *Original Research Article* Pages 47-66.

Lintner, John (1965a,b).The valuation of risk assets and the selection of risky investments in stock Portfolios and capital budgets, *Review of Economics and Statistics*, 47 (1), 13–37.

Markowitz H.M. (1952). Portfolio Selection. *The Journal of Finance*, Vol. 7, No. 1. (Mar., 1952), pp. 77-91.

Markowitz H.M. (1956). The optimization of the quadratic function subject to linear constraints. *Naval Res. Log. Quarterly 3* (1956) 111–133.

Markowitz H.M. (1959). Portfolio Selection, Efficiency Diversification of Investments. *Cowles Foundation Monograph* 16 (Yale University Press, 1959).

Markowitz H.M. (1976). Investment for the Long Run: New Evidence for an Old Rule. The American Finance Association, *The Journal of Finance* Volume 31, Issue 5, pages 1273–1286, December 1976.

Markowitz H.M. (1991) Foundations of Portfolio Theory. *The Journal of Finance* Vol. 46, No. 2 (Jun, 1991), pp. 469-477.

Michaud, R. O. (1998). Efficient Asset Management: A Practical Guide to Stock Portfolio Optimization and Asset Allocation. Boston, MA: *Harvard Business School Press*, 1998.

Michaud, R. O. (2003). An Examination of Resampled Portfolio Efficiency: A Comment by Michaud, R. O. *Financial Analysts Journal*, January/February 2003, Vol. 59, No. 1: 15-16.

Michaud, R. O. (2013). Deconstructing Black–Litterman: How to get the Portfolio you already knew you wanted. *Journal of Investment Management*, Vol. 11, No. 1, (2013), pp. 6–20; © JOIM 2013.

Mossin, Jan. (1966). Equilibrium in a Capital Asset Market, *Econometrica*, Vol. 34, No. 4, pp. 768–783.

Peter C. F. (1977) Mean-Risk Analysis with Risk Associated with Below-Target Returns. The *American Economic Review* Vol. 67, No. 2 (Mar., 1977), pp. 116-126, Published by: American Economic Association.

Sharpe, William F. (1964).Capital asset prices: A theory of market equilibrium under conditions of risk, *Journal of Finance*, 19 (3), 425–442

Theil, H. (1966). Applied Economic Forecasting. *North-Holland Publ. Co*. Amsterdam, 1966.

Treynor, Jack L. (1961). Market Value, Time, and Risk. Unpublished manuscript.

Treynor, Jack L. (1962).Toward a Theory of Market Value of Risky Assets. Unpublished manuscript. A final version was published in 1999, in Asset Pricing and Portfolio Performance: Models, Strategy and Performance Metrics. Robert A. Korajczyk (editor) London: *Risk Books*, pp.15–22.