A random maintenance policy for a repairable system

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RESUMO

Num artigo pioneiro publicado em 1960, Barlow e Hunter consideram um sistema reparável operando sob reparo mínimo e acham a política de manutenção determinística ótima que minimiza o custo esperado por unidade de tempo. Neste trabalho propõe-se uma política aleatória que realiza manutenção cada vez que a intensidade de falha atinge o custo observado por unidade de tempo. Para um sistema que está deteriorando no tempo, mostra-se que a política aleatória tem menor custo esperado que a de Barlow e Hunter para um modelo de reparo geral. No caso particular de reparo mínimo e assumindo um Processo de Lei de Potência, determina-se a distribuição exata do tempo até a manutenção e o custo esperado associado com a política aleatória. Essa distribuição esta relacionada com a Distribuição de Poisson Generalizada.

PALAVRAS CHAVE. Processos de contagem, reparo imperfeito, reparo mínimo.

Tópicos. Estatística, Modelos Probabilísticos

ABSTRACT

In a seminal paper published in 1960, Barlow and Hunter considered a repairable system subject to minimal repairs and found the deterministic optimal maintenance policy which minimizes the expected cost per unit of time. Here it is proposed a random policy which maintains the system whenever the failure intensity exceeds the observed cost per unit of time. When the system is deteriorating over time, it is shown that this random policy has lower expected cost than the periodic one for a general repair model. Moreover, under the minimal repair assumption and assuming a Power Law Process intensity, the exact distribution of the random time and associated cost is shown to be related to the *generalized Poisson distribution*.

KEYWORDS. counting processes, imperfect repair, minimal repair.

Paper topics. Statistics, Probabilistic Models.

1. Introduction

We consider a repairable equipment that is subject to failures at random times. Often, it is assumed that such equipment may undergo two types of actions during operation. On one hand, after each failure, the equipment is *repaired* in order to continue functioning. On the other, at times determined by the operator, it is preventively *maintained* in order to avoid too frequent failures. Operation of these systems should consider the costs of preventive maintenance and of each failure, including here both the cost of the repair action and those due to the system coming unexpectedly to a halt. A *maintenance policy* determines the times at which preventive maintenance should be performed. Designing sound maintenance policies is a main concern in many economic activities.

In the past, much of the engineering literature has focused on models that assume *minimal repair* (MR) and *perfect maintenance* (PM), also known as *as bad as old* (ABAO) and *as good as new* (AGAN) respectively. Systems subject exclusively to MR actions are modeled as a nonhomogenous Poisson process (NHPP), often with a power law intensity $\lambda(t) = (\beta/\eta)\,(t/\eta)^{\beta-1}$. More recently, attention has been given to *imperfect repair* (IR) models, where the effect of each repair leaves the system between the *as bad as old* and *as goos as new* conditions. For instance, Kijima et al. [1988] considered *g-renewal processes* whereby the process begins as an NHPP with a reference intensity $\lambda_R(t)$ but, the effect of each repair after a failure is to reduce the age of the system by a factor of θ, so that the actual intensity of failure at a given moment t is

$$\lambda(t) = \lambda_R[\theta\, t_{N(t)} + t - t_{N(t)}] = \lambda_R[t_{N(t)} - (1-\theta)\, t_{N(t)}]\,, \tag{1}$$

where $t_{N(t)}$ is the last failure before t. Here, t is the *chronological age* while $V(t) = \theta\, t_{N(t)} + t - t_{N(t)} = t_{N(t)} - (1-\theta)\, t_{N(t)}$ is called the *virtual age* of the system. When $\theta = 1$ we have MR because $V(t) = t$ and the process is an NHPP with intensity $\lambda_R(t)$. On the other hand, $\theta = 0$ corresponds to perfect repair because in this case $V(t) = t - t_{N(t)}$ and the process renews after each failure. Hence, θ is called the *efficiency of repair* and measures the amount of rejuvenation introduced by the repair action following a failure. Model (1) is usually called the Kijima virtual age or, following Doyen and Gaudoin [2004], the Arithmetic Reduction of Age of order 1 (ARA_1) model. Alternatively, Lam [1988] introduced the so called *geometric processes*, where the times between successive failures take the form $X_n = Y_n/a^{n-1}$, the Y_ns being *iid* random variables and a a constant that measures the efficiency of the repair actions. While the last word about Imperfect Repair (IR) models has not been written yet, what we are interested here is that, from a mathematical point of view, IR leads to *counting processes* (CPs) with a stochastic intensity $\lambda(t)$ such as (1).

Barlow and Hunter [1960] discussed optimal periodic policies for equipments subject to MR and PM which can be extended in a straightforward manner for the IR case [cf. Kijima et al., 1988; Toledo et al., 2016]. Briefly, suppose that the number of failures $N(t)$ between 0 and t is a counting process (CP) with (possibly random) intensity $\lambda(t)$, and define the unconditional mean function $\Phi(t) = \mathbb{E}\, N(t) = \int_0^t \mathbb{E}\,\lambda(u)\, du$. The *rate of occurrence of failures* (ROCOF) function is $\phi(t) = \Phi'(t) = \mathbb{E}\,\lambda(t)$. It is, essentially, an unconditional version of the intensity $\lambda(t)$. For the MR/NHPP case, the intensity is deterministic and hence $\Phi(t) = \int_0^t \lambda(u)\, du$ and $\phi(t) = \lambda(t)$. The costs associated to the maintenance and repair actions are assumed to be random variables independent of the failure history of the system and having finite expected values $C_M = k$ and $C_R = 1$ (the assumption $C_R = 1$ means that we take the monetary unit to be C_R and carries no loss of generality). Barlow and Hunter's policy is obtained minimizing the expected cost per unit of time $C(t) = [k + N(t)]/t$. Since $\mathbb{E}\, C(t) = [k + \Phi(t)]/t$, defining $B(t) = t\,\phi(t) - \Phi(t)$ and differentiating one obtains that the optimal periodicity τ_{BH} must satisfy

$$\tau_{BH}\,\phi(\tau_{BH}) - \Phi(\tau_{BH}) = k\,. \tag{2}$$

We have assumed here that the ROCOF $\phi(t)$ is nondecreasing and, for τ_{BH} to be finite, that $\lim_{t\to\infty} B(t) > k$. Note that, if $\phi(t)$ is nondecreasing, so should be $B(t)$ because $B'(t) = t\,\phi'(t)$.

The periodic policy τ_{BH} introduced above do not take into account the failure history of the system. However, at least for IR models, the failure history carries a great amount of information about the intensity and, hence, also about the reliability of the system. Therefore, the objective of this paper is to question whether the periodic policy can be improved by taking into account the failure history of the system.

The main idea is quite simple. Suppose as before that $N(t)$ is a CP adapted to the filtration $\mathscr{F}_t$ with intensity $\lambda(t)$. Since for small h we have that $\mathbb{E}[N(t+h) - N(t) \mid \mathscr{F}_t] \approx h\,\lambda(t)$, it follows that

$$\mathbb{E}[C(t+h) - C(t) \mid \mathscr{F}_t] = \mathbb{E}\left[\frac{k+N(t+h)}{t+h} - \frac{k+N(t)}{t} \mid \mathscr{F}_t\right]$$
$$= -\frac{h}{t+h}\,\frac{k+N(t)}{t} + \frac{1}{t+h}\,\mathbb{E}[N(t+h) - N(t) \mid \mathscr{F}_t] \approx \frac{h}{t+h}\,[\lambda(t) - C(t)].$$

Therefore, we expect the cost per unit of time to increase when $\lambda(t) > C(t)$ and to decrease otherwise. This suggests monitoring the intensity $\lambda(t)$ and the cost per unit of time $C(t) = [k + N(t)]/t$ and performing a PM as soon as the former reaches the latter. More precisely, the maintenance policy is defined by the stopping time

$$\tau = \inf\{t > 0 : \lambda(t) \geq C(t)\} \tag{3}$$

(see Figure 1).

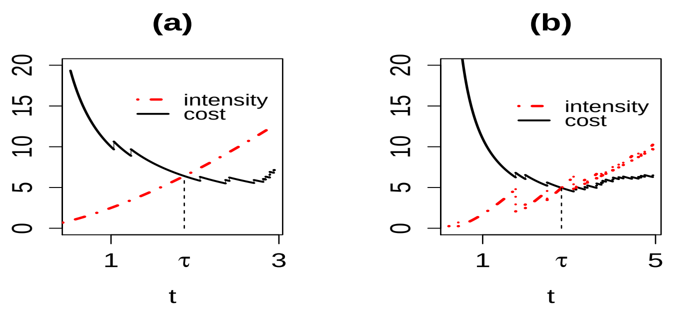

Figura 1: Determination of τ for $k = 10$ and one simulation of (a) an NHPP with intensity $\lambda(t) = (2.5)\,t^{1.5}$ and (b) an ARA$_1$ model with $\lambda_R(t) = (2.5)\,t^{1.5}$ and $\theta = 0.5$.

Before comparing the two policies τ_{BH} and τ, we should discuss the meaning of a *deteriorating* system for IR models. For MR, the process is an NHPP with a deterministic intensity and it is clear that *deteriorating* means that the intensity is nondecreasing. However, for IR, the intensity is subject to random fluctuations and hence the concept is not straightforward. Gilardoni et al. [2016] argue that the appropriate definition, which they call *continuous wear-out* (CWO), should require that the "conditional ROCOF"maps $t \mapsto \mathbb{E}[\lambda(t) \mid \mathscr{F}_s]$ $(t > s)$ be nondecreasing with probability one for all s. They further show that this is equivalent to the requirement that the intensity process $\lambda(t)$ is a submartingale.

Under this perspective, the main results of this paper are the following:

Theorem 1. *Let $N(t)$ be a right-continuous counting process with respect to the filtration $\mathscr{F}_t$ and assume that the intensity $\lambda(t)$ is a submartingale and $\lim_{t\to\infty} B(t) > k$. Then, for τ_{BH} and τ defined respectively in (2) and (3), we have that $\mathbb{E}\,C(\tau_{BH}) \geq \mathbb{E}\,C(\tau)$.*

Theorem 2. *Let $N(t)$ be an NHPP with intensity $\lambda(t) = \beta\,t^{\beta-1}/\eta^{\beta}$ with $\beta > 1$ and define $\mu = \beta^{-1}$ and $a_n = \eta\,[(k+n)/\beta]^{1/\beta}$. Then, for $n \geq 0$,*

$$\mathbb{P}\,[\tau = a_n] = \mathbb{P}\,[N(\tau) = n] = e^{-k\mu}\,k\,\frac{(n+k)^{n-1}}{n!}\,\mu^n\,e^{-n\mu}\,. \tag{4}$$

Theorem 1 states that, for any failure process satisfying the CWO property, the random policy τ outperforms the periodic one τ_{BH}. To our surprise, this holds even for the MR/NHPP case. Next, note that in the MR/NHPP case, since $\tau\,\lambda(\tau) = N(\tau)+k$, the distribution of τ [hence also that of $C(\tau)$] is discrete (cf. Figure 2). Theorem 2 gives an explicit expression for that distribution in the Power Law Process (PLP) case. We note that the distribution (4) is usually known as *Generalized Poisson Distribution* with parameters $\alpha = k\mu$ and $\lambda = \mu$ and has appeared in several other areas [cf. Consul, 1989].

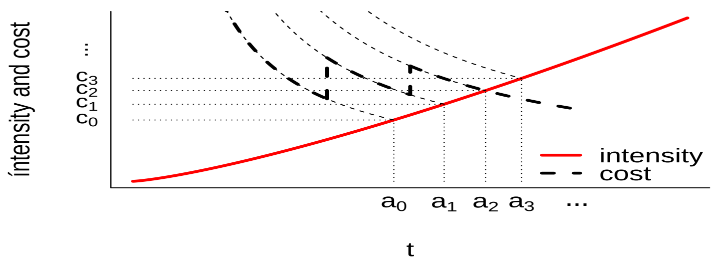

Figura 2: Admissible values of τ (a_n) and $C(\tau)$ (c_n) in the MR/NHPP case. Dashed curves are the maps $t \mapsto (k+n)/t$ ($n = 0, 1, \ldots$). Superimposed there is a trajectory of the cost for which $\tau = a_2$, i.e., the number of failures before maintenance is $N(\tau) = 2$.

The rest of the paper is organized as follows. The next Section contains the proof of Theorem 1. The proof of Theorem 2 is based on optional stopping of Wald's exponential martingale $S_b(t) = \exp\{bN(t) - (e^b - 1)\Lambda(t)\}$ and a connection with Lambert's W function [cf. Corless et al., 1996] which allows us to find the probability generating function of $N(\tau)$. However, since the martingale $S_b(t)$ is not uniformly integrable, the proof becomes somewhat technical and, for reason of space, is not presented here. Notwithstanding, in Section 3 we use the result to compare both $\mathbb{E}\tau$ with τ_{BH} and $\mathbb{E}C(\tau)$ with $\mathbb{E}C(\tau_{BH})$ for several combinations of k and β.

2. Notation and main result

We refer to books such as Kannan [1979] for fundamental results about CPs, stopping times and optional stopping. In what follows all integrals are to be taken in the Stieltjes sense, more precisely $\int_a^b f(u)\,du = \int_{(a,b]} f(u)\,du$. We will assume that all processes and filtrations are right continuous and properties such as stopping times and predictability are with respect to the filtration $\{\mathscr{F}_t : t \geq 0\}$. Whenever possible we will omit dependence on elements of the sample space Ω, so that we will write for instance $X(t)$ instead of $X(t)(\omega)$ and, if σ is a stopping time, $X(\sigma)$ instead $X(\sigma(\omega))(\omega)$. With respect to the counting process $N(t)$, we will assume that it admits a Doob-Meyer decomposition of the form

$$N(t) = \Lambda(t) + M(t) = \int_0^t \lambda(u)\,du + M(t) \tag{5}$$

such that (i) the intensity $\lambda(t)$ is a non negative predictable process such that the maps $t \mapsto \lambda(t)$ are piecewise continuous and locally bounded and (ii) the martingale $M(t)$ is such that $M(0) = 0$ and the maps $t \mapsto M(t)$ have bounded variation and are uniformly integrable.

Lemma 1. *Assume that $N(t)$ is a CP as before and that the intensity process $\lambda(t)$ is a submartingale. Then the process $A(t) = t[\lambda(t) - C(t)] = t\lambda(t) - k - N(t)$ is a submartingale with respect to the filtration $\mathscr{F}_t$.*

Demonstração. Note that $A(t) = \int_0^t u\,d\lambda(u) - k - M(t)$. Now, since $g(u) = u$ is non negative and predictable, the process $\int_0^t u\,d\lambda(u)$ is a submartingale. This and the fact that $M(t)$ is a martingale completes the proof. □

Lemma 2. *Assume that $N(t)$ is a CP with a submartingale intensity $\lambda(t)$. Then, for any two stopping times $\delta \le \sigma$ which are bounded away from zero, in the sense that there exists an $a > 0$ such that $\mathbb{P}(\sigma \ge \delta \ge a) = 1$, we have that*

$$\mathbb{E}[C(\sigma) - C(\delta)] = \mathbb{E}\left(\int_\delta^\sigma \frac{A(u)}{u^2}\, du\right). \tag{6}$$

Demonstração. Since the map $t \mapsto t^{-1}$ is continuous for $t \ge a$, using integration by parts for *càdlàg* processes we have that

$$C(\sigma) - C(\delta) = \frac{k + N(\sigma)}{\sigma} - \frac{k + N(\delta)}{\delta} = \int_\delta^\sigma \frac{1}{u}\, dN(u) - \int_\delta^\sigma \frac{k + N(u)}{u^2}\, du$$
$$= \int_\delta^\sigma \frac{\lambda(u)}{u}\, du + \int_\delta^\sigma \frac{1}{u}\, dM(u) - \int_\delta^\sigma \frac{C(u)}{u}\, du = \int_\delta^\sigma \frac{A(u)}{u^2}\, du + \int_\delta^\sigma \frac{1}{u}\, dM(u).$$

Since $Y_a(t) = \int_a^t u^{-1}\, dM(u)$ $(t \ge a)$ is a UI martingale, it follows from the OST that $\mathbb{E}\int_\delta^\sigma u^{-1}\, dM(u) = \mathbb{E}[Y_a(\sigma) - Y_a(\delta)] = 0.$ □

Note that we can write (3) as $\tau = \inf\{t > 0 : A(t) \ge 0\}$. Therefore, $A(u) < 0$ whenever $u < \tau$ and, using the right-continuity of $A(t)$, $A(\tau) \ge 0$ (see Figure 1). Using this fact we can prove now Theorem 1.

Proof of Theorem 1. It follows from (6) that

$$\mathbb{E}[C(\tau_{BH}) - C(\tau)] = -\mathbb{E}\left(\int_{\tau_{BH}\wedge\tau}^{\tau} \frac{A(u)}{u^2}\, du\right) + \mathbb{E}\left(\int_{\tau}^{\tau_{BH}\vee\tau} \frac{A(u)}{u^2}\, du\right).$$

The first term is nonnegative because $A(u) < 0$ for $u < \tau$. For the second term, note that (i) $\mathbb{1}_{\{\tau_{BH} > \tau\}}$ is $\mathscr{F}_\tau$-measurable and (ii) it follows from Lemma 1 that $\mathbb{E}[A(u) \,|\, \mathscr{F}_\tau] \ge A(\tau) \ge 0$ on the event $\{u \ge \tau\}$. Therefore,

$$\mathbb{E}\left(\int_\tau^{\tau_{BH}\vee\tau} \frac{A(u)}{u^2}\, du \,|\, \mathscr{F}_\tau\right) = \mathbb{1}_{\{\tau_{BH} > \tau\}} \cdot \int_\tau^{\tau_{BH}} \frac{\mathbb{E}[A(u) \,|\, \mathscr{F}_\tau]}{u^2}\, du \ge 0.$$

Now the tower property of conditional expectations complete the proof. ■

3. Distribution of τ and $C(\tau)$ in the PLP case

As mentioned before, for reason of space we omit here the proof of Theorem 1. We finish this section comparing $\mathbb{E}\,\tau$ and $\mathbb{E}\,C(\tau)$ with τ_{BH} and $\mathbb{E}\,C(\tau_{BH})$ in the PLP case with $\beta > 1$. Consider first the deterministic optimal periodicity $\tau_{BH} = B^{-1}(k)$. Since for the PLP we have that $B(t) = t\,\lambda(t) - \Lambda(t) = \beta\,(t/\eta)^\beta$, simple algebra shows that $\tau_{BH} = \eta[k/(\beta-1)]^{1/\beta}$, $\mathbb{E}\,C(\tau_{BH}) = \lambda(\tau_{BH}) = \beta\,\eta^{-1}\,[k/(\beta-1)]^{1-1/\beta}$ and $\mathbb{E}\,N(\tau_{BH}) = \Lambda(\tau_{BH}) = k/(\beta-1)$. To compare with corresponding results for τ, we note that computing moments of $N(\tau)$ is not difficult using for instance well known properties of power series distributions. Indeed, $\mathbb{E}\,N(\tau) = k\,\mu\,(1-\mu)^{-1} = k/(\beta-1)$ and $\operatorname{Var} N(\tau) = k\,\mu\,(1-\mu)^{-3} = k\,\beta^2/(\beta-1)^3$. The fact that $\mathbb{E}N(\tau) = \mathbb{E}N(\tau_{BH})$ seems remarkable.

On the other hand, while we have not been able to find explicit expressions for either $\mathbb{E}\,\tau$ or $\mathbb{E}\,C(\tau)$, since both a_n and c_n have polinomial grow while $p_n = \mathbb{P}\,(\tau = a_n) = \mathbb{P}\,[C(\tau) = c_n]$ decays geometrically, given values of η, β and k, it is easy to compute the series $\mathbb{E}\,\tau = \sum_{n=0}^\infty a_n\,p_n$ and $\mathbb{E}\,C(\tau) = \sum_{n=0}^\infty c_n\,p_n$ to any desired precision. We omit the details here, but show in Figure 3 the behavior of $\mathbb{E}\,\tau$ and $\mathbb{E}\,C(\tau)$ for different values of the shape parameter β and the ratio of costs k and compare them with the corresponding expressions for the optimal deterministic time τ_{BH}.

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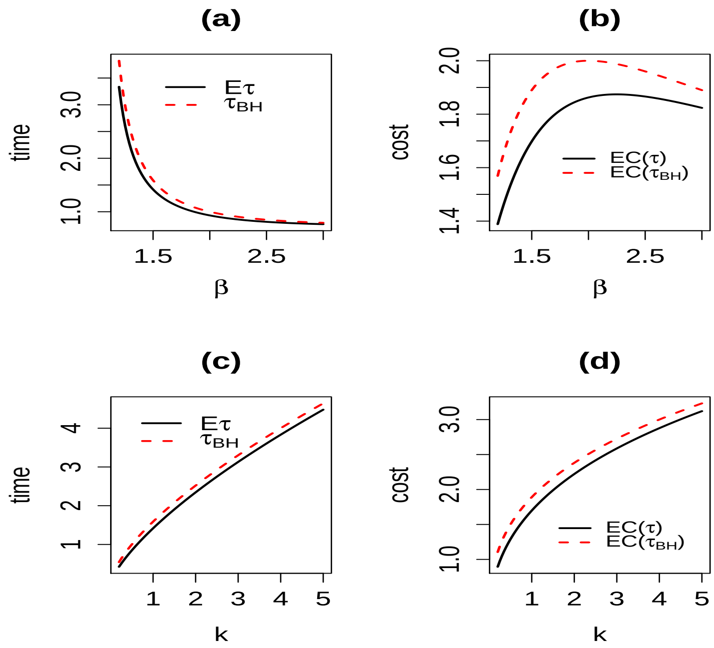

Figura 3: Comparison of $\mathbb{E}\,\tau$ with τ_{BH} and of $\mathbb{E}\,C(\tau)$ with $\mathbb{E}\,C(\tau_{BH})$. Plots (a) and (b) are for a PLP process with $\eta = 1$ and ratio of costs $k = 1$, against the shape parameter β. Plots (c) and (d) are for a PLP process with $\eta = 1$ and shape parameter $\beta = 1.5$, against the ratio of costs k.

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