Optimal Periodic Maintenance Policy under the Arithmetic Reduction Age Imperfect Repair Model

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ABSTRACT

This paper presents the preliminary results of a research work on procedures for the determination of optimal maintenace policies under imperfect repair assumption. The topic was motivated by a real data set on failures of off-road trucks used by a mining company. In particular, it is discussed the determination and practical implementation of an optimal preventive maintenance policy using the ARA-1 (Arithmetic Reduction of Age) model presented by Doyen and Gaudoin (2004). Under such imperfect repair models, the expected number of failures (or mean function) at time t is given by a general renewal function with no closed form solution available. In this work, two procedures to approximate the mean function are proposed. Optimal periodic maintenance policies are obtained for the off-road trucks engines, using the the ARA-1 model and the two proposed approximation procedures.

KEYWORDS: ARA-1 models; optimal maintenace policy; imperfect repair.

MAIN AREA: 1) Probabilistic models; 2) OR in Industry.

RESUMO

Este artigo apresenta os resultados preliminaries de uma pesquisa em procedimentos para a determinação de políticas ótimas de manutenção sob a suposição de reparo imperfeito. O tópico foi motivado por um banco de dados real de falhas em *off-road trucks* utilizados por uma empresa de mineração. Em particular, discute-se a determinação e a implementacão prática de uma política optima de manuenção utilizando o modelo ARA-1 (*Arithmetic Reduction of Age* –Redução Aritmética da Idade), apresentado por Doyen e Gaudoin (2004). Nestes modelos de reparo imperfeito, o número esperado de falhas (or função média) no tempo t é dado por uma *g-renewal function* para a qual não há uma forma fechada. Neste trabalho, dois procedimentos para aproximar a funcão média são propostos. Políticas ótimas de manutencão são obtidas para os motores dos off-road trucks, utilizando o modelo ARA-1 e os dois procedimentos de aproximação propostos.

PALAVRAS CHAVE: modelos ARA-1; política ótima de manutenção; reparo imperfeito.

ÁREA PRINCIPAL: **1**) Modelos Probabilísticos; 2) PO na Indústria.

1. Introduction

There are important references that have proposed maintenance policies to repairable systems, in particular, combining PM and MR. It is fair to say that this combination has been of interest since the work by Barlow and Hunter (1960). The authors used elementary renewal theory to obtain two types of preventive maintenance policies, one which is most useful for simple systems (age replacement policy) and another for complex systems (block replacement policy). The latter proposes to perform minimal repairs (returning the system to an ABAO condition) up to a predetermined time when the system undergoes replacement or PM, returning it to an AGAN condition. The optimal policies obtained depend on the failure distribution, and were proved to be minimal cost solutions. Other results worth mentioning are the ones included in the works by Morimura (1970), Park (1979), Phelps (1981), Barlow and Proschan (1987), Park *et al.* (2000).

Gilardoni and Colosimo (2007) worked on a problem similar to the off-road truck engines but, under a MR environment. The authors applied Barlow and Hunter's block replacement policy in a real data set concerning failures histories of power transformers. They assumed perfect preventive maintenance actions (AGAN) and minimal repairs (ABAO) for failures ocurring between the PM actions. The goal was to find the optimal PM policy given by the check points at every τ units of time. Using a Nonhomogeneous Poisson Process (NHPP) with intensity function $\lambda(t)$ modeled by a Power Law Process (PLP), the authors came up wih an expression of the limiting expected cost per unit of time. Consequently, the optimal PM policy τ can be obtained from a closed form expression, and one needs to have only: (1) the costs ratio ($\frac{C_{PM}}{C_{MR}}$) and (2) estimates for the PLP parameters, obtained from the observed failure data. The authors used Maximum Likelihood estimation and obtained approximate confidence limits for τ.

In other practical situations however, more realistic notions of repair somewhat intermediate between the two extremes AGAN and ABAO might be needed. Many models have already been proposed for imperfect repair effects (for a review see, for example, Pham and Wang (1996)). Among them, are the virtual age models proposed by Kijima *et al.* (1988) and Kijima (1989). In particular, Kijima *et al.* (1988) adapted the block replacement policy by Barlow and Hunter (1960) to the assumption of IR, where the degree of efficiency of the repair is represented by the parameter $\theta(0 \le \theta \le 1)$ and includes ABAO and AGAN as special cases ($\theta = 1$ and $\theta = 0$, respectively). The authors developed a virtual age model to describe the operation in time of a repairable system which is maintaned by an IR. But in this case, as opposed to the MR case developed by Gilardoni and Colosimo (2007), the integral $\Lambda(t) = E\left[N(t)\right] = \int_0^t \lambda(u)du$ which denotes the expected number of failures in the time interval (0,t] does not have a closed form (it is a g-renewal function). To overcame this difficulty, an approximation procedure which can be used to find the optimal replacement periodicity under such conditions was proposed. Still, in the end, the usage of this approximation depends on the knowledge of the repair efficiency (θ value) and the distribution of the lifetime distribution of a new system. Numerical examples were provided by the authors assuming the particular case of a Gamma distribution with given parameter values and different scenarios for cost ratios and repair efficiency. Yet, the model was not statistically studied.

Doyen and Gaudoin (2004) proposed two new classes of imperfect repair models. The repair effect is characterized by the change induced on the failure intensity before and after failure. In the first class of models, repair effect is expressed by a reduction of failure intensity (the so called Arithmetic Reduction of Intensity or ARI models). In the second class, repair effect is expressed by a reduction of the system virtual age (the so called Arithmetic Reduction of Age or ARA models). It is noteworthy that, the virtual age model proposed by Kijima \textit{et al.} (1988) corresponds to a particular case of ARA models, ARA-1. So, from now on, Kijima's model will be refered as ARA-1 in this article. Recently Pan and Rigdon (2009) and Corset *et al.*

(2012) used Bayesian analysis for the ARA and ARI classes of models but the focus was not on optimal maintenance policies. None of these works have already dealt with the problem of the jointly statistical estimation of the repair efficiency and the determination of an optimal maintenance policy.

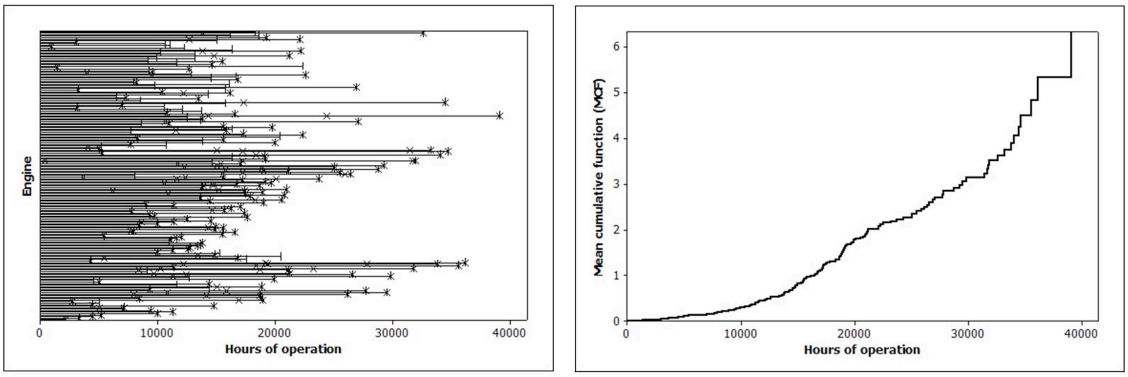
The goal of this research work is to generalize the approach adopted by Gilardoni and Colosimo (2007), now incorporating the class of imperfect repairs of Doyen and Gaudoin (2004), ARA-1. The degree of efficiency of the IR and the optimal PM periodicity parameter (τ) are both estimated using inferential procedures based on the failure history of the systems. In order to make this task possible, a method to estimate $\Lambda(t)$ and $\lambda(t)$ under IR *from the data* is also proposed. The method is applied to the failure histories of off-road engines (see Section 2). In this particular paper we present the preliminary results of the research, namely, the data analysis of the engines. Specifically, we : (1) use the history of failure times of the engines to estimate statistically the degree of efficiency of the imperfect repairs and (2) giving that information, find the optimal maintenance policy under imperfect repairs (OMP-IR), in other words, obtain the optimal PM check points (or periodicity τ) that minimize expected total cost (preventive maintenance + corrective actions) under an environment of imperfect repairs. Other theoretical features of the IR models (i.e. inferential properties, confidence interval coverage studies, etc.) are still under study (see Concluding Remarks and Future Research section).

The outline of the paper is as follows. Section 2 describes the motivating situation. In Section 3, the ARA-1 model is briefly presented along with the cost function. Section 4 deals with statistical methods. In particular, the expression of the likelihood function needed to find the model parameters estimates, namely, the intensity function and efficiency of repair parameters, is derived. Section 5 describes two procedures to approximate the mean function under the ARA-1 model. They are necessary to find the optimal PM policy (given by check points at every τ units of time) . In fact, this is the main contribution of this paper. The proposed procedures are applied to the off-road engines maintenance data and the results are presented in Section 6 (point and interval estimates for τ are provided). Conclusions and final comments end the paper in Section 7.

2. Motivating Situation

This work was motivated by a real situation concerning engine failures on off-road trucks used by a Brazilian mining company. This company keeps a database with detailed descriptions of all maintenance actions performed on their off-road engines. The data used in this paper are a subset of such database, and include preventive (scheduled) and corrective (non-scheduled) maintenance records for a group of 143 diesel engines, for which 208 failure times were recorded. There were 50 preventive maintenance actions during the follow-up period, each assumed to be a perfect repair action, returning the system to as-good-as-new (AGAN) condition. Consequently, the database is consisted, in fact, of $193=143+50$ engines. It is noteworthy that, in this paper, scheduled preventive maintenance (PM) actions, either an overhaul or a replacement by a new system, will be assumed to be a perfect repair.

Figure 1 presents plots of (a) events (failures) vs. operation time (in hours) and (b) the Nelson-Aalen nonparametric estimate (Aalen, 1978) of the mean function $\Lambda(t)$, also known as Mean Cumulative Function (MCF). The convex shape of the MCF (Figure 1-b) indicates that the intensity function of failures is increasing, therefore justifying PM. According to the mining company, the cost of a corrective maintenance performed after a (unexpected) failure is 23% higher than the cost of a preventive maintenance. Hence, the company wants to adopt a maintenance policy that favors preventive maintenance, as opposed to repair actions taken after failures.



(a) (b)

Figure 1: (a) Events (failures) vs. operation time (in hours); (b) Mean Cumulative Function (MCF)

3. Cost function and optimal PM under ARA-1 model

Consider a system which is subject to failure, and that is put in operation at time $t = 0$. Assume the following conditions:

1. PM check points are scheduled after every τ units of time;
2. at each PM check point, a repair action (or a replacement) of fixed cost C_{PM} is executed,which instantly returns the system to an AGAN condition;
3. between successive PM check points, an IR of degree $\theta(0 \le \theta \le 1)$ is done after each failure;
4. the expected cost for each IR action ;
5. repair costs and failure times are independent;
6. repair times are neglected.

Let $N(t)$ be the number of failures in the time interval (0,t]. The PM policy that minimizes the long run expected cost per unit of time is the value of τ that satisfies (Gilardoni and Colosimo, 2007):

$$\tau\lambda(\tau) - \Lambda(\tau) = \frac{C_{PM}}{C_{IR}} \qquad (1)$$

Note that only the ratio between costs must be considered, what simplifies the application in practice. Under the imperfect repair assumption, some functional forms for $\lambda(t)$ have been proposed in literature. ARA-1 model uses the notion of *virtual age* $V_n = \theta T_n$ where T_n is the random variable representing the real age of the system at the n^{th} failure (the elapsed time since the initial start-up of the system), and V_n is the virtual age of the system immediately after the n^{th} repair. If $\theta = 1$, it follows that $V_n = T_n$, in which case it is assumed that a MR is performed. This assumption makes the underlying failure process a NHPP. Futhermore, if $\theta = 0$, then $V_n = 0$, indicating that the system is renewed after each repair and the resulting process is a Renewal Process. The failure intensity function of the system under ARA-1 model is given by:

$$\lambda(t) = \lambda_R(t - (1-\theta)T_{N(t)}) \qquad (2)$$

where $T_{N(t)}$ denotes the elapsed time since the initial start-up of the system and the occurrence of the N^{th} failure and λ_R is the failure intensity function corresponding to the condition of minimal repair (reference function). With the intensity function (2), it can be shown (Kijima *et al.*,1988) that the expected number of failures (or mean function) at time τ is given by

$$\Lambda(\tau) = \int_0^{\tau} E\left[\lambda_R(t-(1-\theta)T_{N(t)})\right]dt \qquad (3)$$

There is no closed form solution for equation (3) , except on the special case of $\theta = 1$ (minimal repair). Approximate methos have been proposed for the general case (3) (Kijima et al., 1988; Yevkin and Krivtsov, 2000). However, none of these Works have proposed a method with desirable statistical properties and capable of using the observed failure histories to deal with the following three issues at the same time, namely: (1) the estimation of the parameters involved in Equation 2, (2) the calculation of an approximation for the mean function $\Lambda(t)$ (Equation 3) and (3) the combination of (1) and (2) to solve Equation 1 for τ. For practical purposes, the first step to find the optimal PM policy is to estimate the model parameters. Section 4 introduces some additional notation and presents the likelihood function for this IR model. In special, inference procedures for the parameters of the Power Law Process.

4. Parameter Estimation: Likelihood Function

The likelihood function is constructed here assuming that among the k observed repairable systems, k_1 are time truncated and k_2 are failure truncated, $k_1, k_2 = 1,2,\ldots,k$ and $k_1 + k_2 = k$. Let μ denote the vector of model parameters. It includes the parameters indexing the process intensity function and the repair efficiency parameter θ. For example, if the PLP is used (Crow, 1974), then, the reference intensity function in Equation 2 and its associated mean function are given, respectively by

$$\lambda_R(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \quad \text{and} \quad \Lambda_R(t) = \int_0^t \lambda_R(u)\,du = \left(\frac{t}{\eta}\right)^{\beta}; \eta, \beta, t > 0 \qquad (4)$$

where β (shape parameter) represents how the system deteriorates or improves over time, and η is a scale parameter. In this case, $\mu = (\beta;\eta;\theta)^t$ (a 3x1 vector). Using the induced intensity function of the ARA-1 model (Equation 2) and the PLP (Equation 4), the loglikelihood function takes the form:

$$\begin{aligned} l(\mu) &= N\log(\beta) - \beta N\log(\eta) + \\ &+(\beta-1)\left(\sum_{i=1}^{k_1}\sum_{l=1}^{n_i}\log(t_{i,l}-(1-\theta)t_{i,l-1}) + \sum_{j=1}^{k_2}\sum_{m=1}^{n_j^*}\log(t_{j,m}-(1-\theta)t_{j,m-1}) \right) + \\ &+\left\{\sum_{i=1}^{k_1}\left[\sum_{l=1}^{n_i}\left(\frac{\theta t_{i,l}}{\eta}\right)^{\beta} - \left(\frac{t_{i,l}-(1-\theta)t_{i,l-1}}{\eta}\right)^{\beta}\right] - \left(\frac{t_i^*-(1-\theta)t_{i,n_i}}{\eta}\right)^{\beta}\right\} + \qquad (5) \\ &+\left\{\sum_{j=1}^{k_2}\left[\sum_{m=1}^{n_j^*}\left(\frac{\theta t_{j,m}}{\eta}\right)^{\beta} - \left(\frac{t_{j,m}-(1-\theta)t_{j,m-1}}{\eta}\right)^{\beta}\right] - \left(\frac{\theta t_{j,n_j^*}}{\eta}\right)^{\beta}\right\} \end{aligned}$$

According to Zhao and Xie (1996), it can be assumed that $\hat{\mu} = \arg\max L(\mu)$ follows approximately a multivariate normal distribution, with mean μ and covariance matrix Σ given by minus the inverse of the Hessian matrix of $l(\mu) = \log L(\mu)$ evaluated at $\hat{\mu}$. The Hessian is given by

$$H = -\left[\frac{\partial^2 l(\mu)}{\partial\mu\partial\mu^t}\right] \quad (6)$$

Hence, asymptotic theory can be used to construct confidence intervals for the parameters.

5. Proposed procedures to obtain the optimal PM.

For practical purposes, it is necessary to estimate the optimal maintenance periodicity τ using the failure history of the systems under study. However, as it was mentioned in Section 2, there is no closed form solution for the g-renewal function given by Equation 3. In this Section, two procedures to estimate the mean and intensity functions ($\Lambda(t)$ and $\lambda(t)$, respectively), *from the data* are proposed. These estimates obtained by each one of the procedures are then used in the cost function (Equation 1) to calculate a point estimate of the PM periodicity parameter τ. Subsequently, confidence intervals for τ are obtained using Bootstrap resampling method. In either one of the procedures, the mean function $\Lambda(t)$ is estimated using a combination of Monte Carlo simulation and the Nelson-Aalen nonparametric procedure (Aalen, 1978), also known as Mean Cumulative Function (MCF). In the sequel, we use the term MCF whenever we refer to the Nelson-Aalen estimate. The steps of the proposed procedures are ilustrated using the PLP but it can be applied to any other parametric form chosen for the (reference) intensity function ($\lambda_R(t)$).

For both procedures, the first step is to obtain the MLE's $\hat{\eta},\hat{\beta}$ (PLP parameters) and $\hat{\theta}$ (repair efficiency) using the observed failure history of the systems under study. Next (step 2) k=10,000 sistems are generated (Monte Carlo Simulation). The failure histories of those systems were generated using the point estimates $\hat{\eta},\hat{\beta}$ (PLP parameters) and $\hat{\theta}$ (repair efficiency) obtained in step 1. From the next step on (step 3) the main diferences among the two procedures are basically the following.

In the first procedure, which will be refered to throughout this paper as ***polinomial approach***, the generated failure histories (k=10,000 systems) are used to find the MCF $\hat{\Lambda}(t)$ and a second order polinomial is fitted (by least squares) to the points $(\hat{\Lambda}(t);t)$ in order to approximate this step function. The first derivative leads to the funcitonal form of $\hat{\lambda}(t)$. In the second procedure, which will be refered to thoughout this paper as ***GCM (Greatest Convex Minorant) approach,*** one first applies a TTT-transformation (Total Time on Test transformation) to the the generated failure histories of the k=10,000 systems, then constructs the MCF estimate $\hat{\Lambda}$. Following Gilardoni and Colosimo (2011), an estimate $\hat{\lambda}(t)$ of $\lambda(t)$, which takes into account the monotonicity constraint can now be obtained as the derivative of the greatest convex minorant of the MCF estimate $\hat{\Lambda}$. The TTT-transformation is necessary in order to make sure that the good properties of the estimate are preserved (see Gilardoni and Colosimo, 2011 for details).

In both cases, a point estimate for the optimal PM periodicity parameter τ (namely $\hat{\tau}_{pol}$ and $\hat{\tau}_{GCM}$) is obtained using given cost ratios and solving Equation 1 for τ.

5.1 Confidence intervals

Appropriate confidence intervals for the optimal PM periodicity are obtained using nonparametric Bootstrap resampling method (Efron and Tibshirani, 1986). The method consists of resampling with replacement *B* (*B* large) samples from the original database. Each resample has the same size as the original data set. For the off-road engines, for example, the database consists of 193 engines. Thus, each one of the *B* resamples has 193 engines resampled with replacement from the original database. Here, we use *B=1,000*. The two procedures (polinomial and GCM) described previously are then applied to each one of the *B* resamples, thereby obtaining *B* bootstrap estimates for τ, denoted here by $\tau^*_{pol_i}(i=1,\ldots,B)$ and $\tau^*_{GCM_i}(i=1,\ldots,B)$ (polinomial and GCM procedures, respectively). The $(1-\alpha)\times100\%$ CI for τ, given by the polinomial procedure is then given by the limits $(\hat{\tau}^*_{pol[l]};\hat{\tau}^*_{pol[u]})$, where $\hat{\tau}^*_{pol[i]},(i=1,\ldots,B)$ are the bootstrap estimates obtained by the polinomial procedure, sorted in increasing order, $l=B\times(\alpha/2)$ and $u=B\times(1-\alpha/2)$, l and u rounded to the smallest and largest nearest integers, respectively. The confidence interval based on the GCM method is obtained similarly.

6. Motivating situation revisited

The analyses were done using a script written in R, a language and environment for statistical computing (www.R-project.org, v.2.15). The goal here is twofold (1) estimate the degree of efficiency of repairs and (2) obtain the optimal PM check points that minimize expected total cost.

We start with parameter estimation by finding the MLEs of the PLP parameters (β and η) and the degree of repair parameter θ. Next, we use the two procedures briefly described in Section 5 and get point estimates for the periodicity parameter τ along with 95% (bootstrap) confidence intervals.. Finally, we compare the results under IR to the ones obtained assuming PLP and MR, as proposed by Gilardoni and Colosimo (2007).

Under the general assumption of IR, a ARA-1 model was applied and the likelihood function described in Section 4 was used to obtain the following MLEs and CIs for the parameters: $\hat{\beta}=2.458(2.185;2.765), \hat{\eta}=15{,}585(14{,}605;16{,}633)$ and $\hat{\theta}=0.471(0.330;0.673)$. Note that the estimated value for θ and the corresponding confidence interval suggest that the repair actions after failures are neither minimal ($\theta=1$) nor perfect repairs ($\theta=0$). Therefore, the traditional modeling assumptions are inappropriate for the off-road engines, reassuring the relevance of the methodology proposed in this paper.

The two procedures described in Section 5 were used to estimate the optimal maintenance time under the IR assumption(namely $\hat{\tau}_{pol}$ and $\hat{\tau}_{GCM}$), using $K=10{,}000$ and $B=1{,}000$. Also, a truncation time $T=40{,}000$ h was used; that corresponds to the time range in the observed data. For the polynomial procedure, the fitted curve was $\hat{\tilde{\Lambda}}(t)=8\times10^{-6}t+2.9\times10^{-9}t^2$ and seemed to provide a good approximation for the step function.

Recall that according to the mining company, the cost of a corrective maintenance performed after a (unexpected) failure is 23% higher than the cost of a preventive maintenance. Therefore, point estimates (and confidense intervals) for the optimal PM periodicity parameter τ were obtained for the ratio value $C_{IR}/C_{PM}=1.23$ using the polinomial ($\hat{\tau}_{pol}$) and the GCM procedure ($\hat{\tau}_{GCM}$). In addition, an estimated value was also obtained using the modeling approach presented by Gilardoni and Colosimo (2007), i.e., assuming PLP and MR. Although asymptotic theory can be used to construct confidence intervals under the MR assumption, bootstrap confidence intervals were also obtained for that case, for the sake of comparison. The resultas are summarized in Table 1 (this table also includes the estimated values for two other cost ratio values).

Table 1 Optimal Periodic Maintenance Policy by Cost Ratio (C_{IR}/C_{PM}), under MR ($\hat{\tau}_{MR}$) and IR assumption - ARA-1 model : $\hat{\tau}_{pol}$ (polinomial procedure) and $\hat{\tau}_{GCM}$ (GCM procedure).

C_{IR}/C_{PM}	$\hat{\tau}_{MR}$ (h)	$\hat{\tau}_{pol}$ (h) (ARA-1, polinomial)	$\hat{\tau}_{GCM}$ (h) (ARA-1, GCM)
1.23	14,345 (13,271;15,474) [a]	16,704 (13,822 ; 20,297)	15,709 (13,294;17,585)
3	9,430 (8,877;9,982)	10,696 (8,859;13,033)	8,520 (8,360,10,043)
10	5,352 (4,939;5,810)	5,858 (4,860;7,128)	5,274 (4,640,6,098)

a: 95% C.I – non-parametric Bootstrap

The main observations from Table 1 are:

1. Using the cost ratio provided by the mining company (1.23) the optimal maintenance periodicity obtained under the ARA-1 model was $\hat{\tau}_{pol} = 16,704$ hours or 696 days (using the polynomial procedure) and $\hat{\tau}_{GCM} = 15,709$ hours or 654 days (using the GCM procedure). Note that both values are larger than the one provided under the MR assumption (inappropriate for this practical situation).
2. The values of the point estimates of the optimal periodic maintenance parameter τ decrease with the increase of the cost ratio, no matter which assumption is made about the repairs (i.e, MR or IR-ARA-1) .This result was already expected since it is better to implement a PM before failures occur when the cost of a repair is too high. In particular, the three point estimated values are pretty much the same for the highest cost ratio illustrated here (10).
3. As far as the ARA-1 model, the GCM procedure seems to provide smaller point estimates for the parameter τ, although the difference decreases with the increase of the cost ratio value.

7. Concluding remarks and future work

In this paper, an optimal preventive maintenance (PM) check point was obtained for the case of repairable systems subject to perfect preventive maintenance actions (which returns them to an AGAN condition) and imperfect repairs (IR) after a failure. The IRs were assumed to be of degree θ ($0 \leq \theta \leq 1$, unknown), following an ARA-1 model. The motivating practical situation concerned failures histories of off-road truck engines used by a mining company. Model parameters estimates were jointly obtained by the Maximum Likelihood method, namely, the PLP parameters and the degree of repair θ.

Next, two procedures for estimating the mean function $\Lambda(t)$ in an ARA-1 model were presented. The methods combined Monte Carlo simulation, the calculation of the (nonparametric) Mean Cumulative Function and aproximations of the MCF using a second order polinomial (polinomial procedure) and the GCM. Those procedures made it possible to estimate the optimal preventive maintencance check point (τ) for the practical situation under study. Confidence intervals were also obtained for this quantitiy, using Bootstrap resamping method. The results were compared with the ones obtained under a minimal repair assumption. Recall that, by using the ARA-1 type model, the main challenge here was not only to come up with good approximations to the g-renewal function involved in the expression of the Mean Function ($\Lambda(t)$) but one differentiable. With this in mind, one solution was to fit a polynomial curve to the MCF points and the other, to use the GCM. Point estimates based on the two procedures were compared. Nevertheless, it is important to find a better way to compare those results. Note that it is not possible to compare them via a Monte Carlo Simulation for example (using diferent values of the PLP and the repair efficiency parameter θ) as it is usually done in statistical studies, since we cannot calculate the "real value" of Equation 3. (the g-renewal function). This difficulty was exactly what motivated this research in the first place.

What can be done is to use both methods in a situation where it is possible calculate the real valuses, for instance the MR case. That is one of the follow ups of this research. Another follow-up is to verify via simulation, the real coverage of the intervals provided by each one of the procedures (polinomial and GCM) and the average lenth of those intervals.

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