On the sum of the two largest signless Laplacian eigenvalues

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Abstract

Let G be a simple graph on n vertices and m edges. Consider $L(G) = D - A$ and $Q(G) = D + A$ as the Laplacian and the signless Laplacian of G, where A is the adjacency matrix and D is the diagonal matrix of the vertices degree of G. Brouwer conjectured that the sum of the k largest Laplacian eigenvalues of G is at most $m + \binom{k+1}{2}$. Haemers *et. al.* in 2010 proved that this result is valid for $k = 2$. In this paper, we investigate this problem for the signless Laplacian matrix when $k = 1$ and $k = 2$.

Keywords: *signless Laplacian; sum of eigenvalues; bounds.*

1 Introduction

Given a simple graph G with vertex set $V(G)$ and edge set $E(G)$, write A for the adjacency matrix of G and let D be the diagonal matrix of the row-sums of A, i.e., the degrees of G. The maximum degree of G is denoted by $\Delta = \Delta(G)$. Let $e(G) = |E(G)|$ be the number of edges and let $n = |V(G)|$ be the number of vertices of G. If H is a subgraph of G, we write n_H for the number of vertices of H. The matrix $Q(G) = A + D$ is called the *signless Laplacian* or the Q-matrix of G. As usual, we shall index the eigenvalues of $Q(G)$ in non-increasing order and denote them as $q_1 \geq q_2 \geq \ldots \geq q_n$. The Laplacian matrix of G is given by $L(G) = D - A$ and its eigenvalues are also arranged in non-increasing order and we denote them as $\mu_1 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$. We denote $\overline{G}$ as the complement graph of G, and denote K_n, C_n, S_n as the complete, cycle and star graphs on n vertices.

Consider $M(G)$ as the adjacency, Laplacian or signless Laplacian matrix of a graph G of order n and let k be a natural number such that $1 \leq k \leq n$. A general question related to G and $M(G)$ can be raised: *"How large can be the sum of the k largest eigenvalues of $M(G)$?"*

In [6], Ebrahimi *et al.*, bounded the sum of the two largest eigenvalues of the adjacency matrix. In [9], Haemers, Mohammadian and Tayfeh-Rezaie presented Brouwer's conjecture for the sum of the k largest eigenvalues of the Laplacian matrix.

Conjecture 1.1 *Let G be a graph on $e(G)$ edges. Then,*

$$S_k(G) = \sum_{i=1}^{k} \mu_i(G) \leq e(G) + \binom{k+1}{2}. \tag{1}$$

Haemers, Mohammadian and Tayfeh-Rezaie, [9], solved Conjecture 1.1 for every k when G is a tree and also for every graph G when $k = 2$. More recently, Du and Zhou [3] proved that the conjecture is true for unicyclic and bicyclic graphs. It turns out that the same upper bound of the Conjecture 1.1 seems to be true to the sum of the k largest eigenvalues of the signless Laplacian of a graph G, denoted by $T_k(G)$. We state that as a conjecture and it drives our motivation throughout this paper.

Conjecture 1.2 *Let G be a graph on $e(G)$ edges. Then,*

$$T_k(G) = \sum_{i=1}^{k} q_i(G) \leq e(G) + \binom{k+1}{2}.$$

Observe that Conjecture 1.2 is true for every simple graph G when $k = n$ and $k = n-1$. It is possible to determine some classes of graphs that satisfy Conjecture 1.2 for $k = 2$. See for instance the regular graphs. If G is $r-$regular, then $q_i(G) = 2r - \mu_{n-i+1}(G)$ for each $i = 1, 2, \cdots, n$. Thus, for $n \geq 8$, $T_2(G) = 4r - \mu_{n-1}(G) \leq e(G) + 3$, since for $n \geq 8$, $4r - \mu_{n-1}(G) \geq e(G) + 3$ if and only if $2\mu_{n-1}(G) \leq (8-n)r - 6 < 0$, which implies that $\mu_{n-1} < 0$, and it is a contradiction. However, the proof of the general conjecture is not trivial. In this paper, we devote our attention to prove the cases: $k = 1$ for any graph G and $k = 2$ to the unicyclic graphs.

Moreover, from the Conjectures 1.1 and 1.2, one can raise the following question: *is that possible to compare S_k and T_k?* It is known that $q_1(G) \geq \mu_1(G)$, [1], and so $T_1(G)$ is always greater than or equal to $S_1(G)$. For $k = n$, $T_n(G) = S_n(G) = 2m$. However, if we take the complete graph K_5 and the cycle graph of 5 vertices plus one edge as G_1, we obtain $T_2(K_5) > S_2(K_5)$ and $T_2(G_1) < S_2(G_1)$, and then for $k = 2$, S_2 and T_2 are incomparable. Therefore, we cannot guarantee that S_k is bounded above by T_k for $k = 3, \ldots, n-1$. This fact shows that finding upper bounds to these two parameters can be relevant.

2 Preliminary results

Let us consider a Hermitian matrix A and its eigenvalues as $\lambda_1(A), \ldots, \lambda_n(A)$ arranged in non-increasing order. Recall that Ky Fan, in [7], proved an interesting inequality relating the sum of the eigenvalues of two symmetric matrices, A and B, to the eigenvalues of the matrix $A+B$. That result is important for our purposes in this paper and we shall use it in order to prove our main result, Theorems 3.8.

Theorem 2.1 **([7])** *Let A and B be two real symmetric matrices of size n. Then for any $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B),$$

where, for a matrix M, $\lambda_i(M)$ denotes the largest i-th eigenvalue of M.

From Ky Fan theorem we prove Propositions 2.2 and 2.3 as it has been done by Du and Zhou in [3] for the Laplacian matrix.

Proposition 2.2 **([3])** *Let H be a subgraph of a graph G and $n_H \geq 2$ vertices. Then*

$$T_k(G) \leq T_k(H) + 2(e(G) - e(H))$$

for $1 \leq k \leq n_H$.

Proposition 2.3 **([3])** *Let G be a graph with $e(G)$ edges and maximum degree $\Delta \geq 2$. Then*

$$T_k(G) \leq 2e(G) - \Delta(G) + k$$

for $1 \leq k \leq \Delta - 1$.

Following, we present an upper bound to $T_k(G)$ as a function of the clique number of G.

Proposition 2.4 *Let G be a graph with clique number $\omega \geq 3$. Then*

$$T_k(G) \leq 2e(G) - 2k + \omega\,(k + 2 - \omega)\,,$$

for $1 \leq k \leq \Delta - 1$.

Proof From Proposition 2.2 follows

$$\begin{aligned} T_k(G) &\leq T_k(K_\omega) + 2\left(e(G) - \binom{\omega}{2}\right) \\ &= k\omega + \omega - 2k + 2\left(e(G) - \binom{\omega}{2}\right) \\ &= 2e(G) - 2k + \omega(k + 2 - \omega) \end{aligned}$$

■

Let G_1 and G_2 be vertex disjoint graphs. We denote by $G_1 \sim G_2$ a graph obtained from G_1 and G_2 by connecting a vertex of G_1 to a vertex of G_2 with an edge. Also, let $G_1 \approx G_2$ the graph obtained from G_1 and G_2 by inserting two edges between $V(G_1)$ and $V(G_2)$. The next two lemmas prove that Conjecture 1.2 is true for the graphs resulting from those operations if the conjecture is true for G_1 and G_2. The proof follows from Lemma 2.6 and Lemma 2.7 due to Wang, Huang and Liu in [11].

Lemma 2.5 *Let G_1 and G_2 be two graphs of order n_1 and n_2 and size $e(G_1)$ and $e(G_2)$, respectively. If $e(G_i) \geq 1$ and $T_{k_i}(G_i) \leq e(G_i) + \binom{k_i+1}{2}$ for $k_i = 1, 2, \cdots, n_i$ and $i = 1, 2$, then for $1 \leq k \leq n_1 + n_2$,*

$$T_k(G_1 \sim G_2) \leq e(G_1 \sim G_2) + \binom{k+1}{2}.$$

Lemma 2.6 *Let G_1 and G_2 be two graphs of order n_1 and n_2, respectively. If $e(G_i) \geq 2$ and $T_{k_i}(G_i) \leq e(G_i) + \binom{k_i+1}{2}$ for $k_i = 1, 2, \cdots, n_i$ and $i = 1, 2$, then for $1 \leq k \leq n_1 + n_2$,*

$$T_k(G_1 \approx G_2) \leq e(G_1 \approx G_2) + \binom{k+1}{2}.$$

3 Main results

We begin this section proving Conjecture 1.2 for any graph when $k = 1$.

Theorem 3.1 *Let G be a graph of size $e(G)$. Then*

$$T_1(G) = q_1(G) \leq e(G) + 1.$$

Equality holds if and only if G is isomorphic to S_n.

Proof Consider G as graph on n vertices and $e(G)$ edges. We shall prove the theorem in two parts: in (A) we assume G is connected and in (B) G is disconnected. Let us start proving part (A). It is easy to check that all connected graphs on $1 \leq n \leq 4$ satisfy $q_1 \leq e(G) + 1$. As proved in [10] and [4], $q_1(G)$ of connected graphs on $n \geq 5$ is bounded above by

$$q_1(G) \leq \frac{2e(G)}{n-1} + n - 2, \tag{2}$$

with equality if and only if G is isomorphic to K_n or S_n.

Using inequality (2) and considering $n \geq 5$, we get

$$\begin{aligned} q_1 - (e(G)+1) &\leq \frac{2e(G)}{n-1} + n - 2 - (e(G)+1) \\ &= \frac{2e(G) + (n - e(G) - 3)(n-1)}{n-1} \\ &= \frac{2e(G) + (n - e(G) - 1)(n-1) - 2(n-1)}{n-1} \\ &= \frac{2(e(G) - n + 1) + (n - e(G) - 1)(n-1)}{n-1} \\ &= \frac{(n - e(G) - 1)(n-3)}{n-1} \\ &\leq 0. \end{aligned}$$

This proves the part (A) of the theorem. Now, consider that G is disconnected and has at leat two connected components. Assume that the index of G comes from a component G_i of G, say G_1, with $e(G_1)$ edges. Applying the result obtained at part (A) to this connected component, we have $q_1 \leq e(G_1) + 1 \leq e(G) + 1$. It proves the part (B) of the theorem. Equality case is obtained from equality conditions to the inequality (2) and it completes the proof of the theorem. ■

We checked Conjecture 1.2 for all graphs with at most seven vertices when $k = 2$ and the following lemma is stated as a result of the computational experiments.

Lemma 3.2 *If G is a graph of order $n \leq 7$ and size m then $T_2(G) \leq e(G) + 3$.*

It is easy to see that if Conjecture 1.2 holds to disconnected graphs, it also holds for connected graphs. The proof follows from Wang, Huang and Liu, [11], in Lemma 2.2.

In [12], Yan proved that if G is a graph on $n \geq 2$ vertices, then $q_1(G) \leq 2n-2$ and $q_2(G) \leq n-2$. Also, Yan also proved that the complete graphs are extremal to the first upper bound but are not the only ones. Recently, de Lima and Nikiforov, [2], characterized all graphs for which $q_2(G)$ is equal to $n-2$. Therefore, a natural upper bound to $q_1(G)+q_2(G)$ of a graph G is $3n-4$ and the Conjecture 1.2 is true for graphs on $n \geq 2$ which the number of edges $e(G)$ are at most $3n-7$. Moreover, since $T_n(G) = 2e(G)$, it is reasonable to think that dense graphs satisfies Conjecture 1.2 and this is proved in the next result.

Lemma 3.3 *Let G be a connected graph of order $n \geq 5$ and size $e(G) \geq 2n-3$. Then $T_2(G) \leq e(G)+3$.*

Proof Consider $e(G) \geq 2n-1$. Since $q_1 \leq \frac{4m}{n} + n - 4 + \frac{4}{n}$, as proved in [8], and $q_2(G) \leq n-2$, it follows that

$$\begin{aligned} T_2(G) - (e(G)+3) &\leq \frac{4e(G)}{n} + n - 4 + \frac{4}{n} + n - 2 - e(G) - 3 \\ &= \frac{-e(G)n + 4e(G) + 2n^2 - 9n + 4}{n} \\ &= \frac{-e(G)(n-4) + (2n-1)(n-4)}{n} \\ &= \frac{(n-4)(2n-e(G)-1)}{n} \\ &\leq 0. \end{aligned}$$

Now, let us consider $e(G) \in \{2n-3, 2n-2\}$. Using $q_1(G) \leq \frac{2e(G)}{n-1} + n - 2$ that can be obtained from [10] and [4], it follows that

$$\begin{aligned} T_2(G) - (e(G)+3) &\leq \frac{2e(G)}{n-1} + n - 2 + n - 2 - e(G) - 3 \\ &\leq 2n - e(G) - 3 \\ &\leq 0, \end{aligned}$$

and the proof is completed. ■

Let H be a subgraph of G. We shall write $G \backslash H$ for the subgraph obtained by removing the edges of H.

Lemma 3.4 *If G is a unicyclic graph of order n and girth $g \in \{4, 6\}$ or $g \geq 8$, then $T_2(G) \leq e(G) + 3$.*

Proof Firstly, if G is an unicyclic graph with even girth, then G is bipartite. Since the Laplacian and signless Laplacian spectrum coincides, using the result proved by Haemers *et al.* in [9] the result follows.

Hence, consider G as an unicyclic graph with odd girth g and denote the induced cycle by C_g such that $e(C_g) = g$. It is well-known that $q_1(C_g) = 4$ and $q_2(C_g) = 2 + 2\cos(\frac{2\pi}{g})$ and then $T_2(C_g) = 6 + 2\cos(\frac{2\pi}{g})$. The graph $G \setminus C_g$ obtained by removing the edges of C_g from G is bipartite and from Haemers *et al.*, it also satisfies $T_2(G \setminus C_g) \leq n - g + 3$.

From Theorem 2.1,

$$\begin{aligned} T_2(G) &\leq T_2(C_g) + T_2(G \setminus C_g) \\ &\leq \left(6 + 2\cos(\frac{2\pi}{g})\right) + n - g + 3 \\ &\leq (8 - g) + (n + 3). \end{aligned}$$

Thus, for $g \geq 8$, we get $T_2(G) \leq n + 3 = e(G) + 3$ and the result follows. ■

Lemma 3.5 *If G is a unicyclic graph of order n without pendant edges attached to the vertices of the cycle then $T_2(G) \leq e(G) + 3$.*

Proof Let C_g be the cycle of G with order g and let $G - C_g$ be the graph obtained from G by removing the vertices of the cycle C_g. So, $G - C_g$ has $1 \leq t \leq p$ connected components denoted by $H_1, \ldots, H_p$ and we can write G isomorphic to $(((C_g \sim H_1) \sim H_2) \sim \cdots \sim H_p)$. From Lemma 2.5, the results follows since $T_2(C_g) \leq e(C_g) + 3$ and $T_2(H_i) \leq e(H_i) + 3$ for each $i = 1, \ldots p$. ■

Lemma 3.6 *Let G be a unicyclic graph of order $n \geq 4$ with girth $g \in \{3, 5, 7\}$ and $n - g$ pendant vertices. Then $T_2(G) \leq e(G) + 3$.*

Proof Let C_g be the cycle of G induced by the vertex set $V(C_g) = \{u_1, \ldots, u_g\}$ and each vertex $u_i \in V(C_g)$ has $r_i \geq 0$ pendant vertices for $i = 1, \ldots, g$. Our proof consider the following three cases.
Case (A): Assume $g = 3$ and let $V(C_3) = \{u_1, u_2, u_3\}$.

(*i*) Consider that $r_1 = n - 3$ and $r_2 = r_3 = 0$. From Theorem 2.1, $T_2(G) \leq T_2(S_n) + T_2(K_2) \leq n + 3 = e(G) + 3$;

(ii) Consider that $r_1 = 0$ and $r_2 \geq r_3 \geq 1$. Define G_1 as the star S_{r_2+1} rooted in u_2 and G_2 as the star $S_{r_3+1} \sim H$ rooted in u_3 such that H is the subgraph of G induced by the vertex u_1. From Lemma 2.6, $T_2(G) = T_2(G_1 \approx G_2) \leq e(G_1 \approx G_2)+3 = e(G)+3$;

(iii) Consider that $r_1 \geq r_2 \geq r_3 \geq 1$. The subgraph $G \setminus C_3$ is isomorphic to $S_{r_1+1} \cup S_{r_2+1} \cup S_{r_3+1}$. From Theorem 2.1, $T_2(G) \leq T_2(C_5) + T_2(S_{r_1+1} \cup S_{r_2+1} \cup S_{r_3+1}) = 5 + r_1 + 1 + r_2 + 1 < e(G) + 3$.

Case (B): Assume $g = 5$ and let $V(C_5) = \{u_1, u_2, u_3, u_4, u_5\}$.

(i) Consider that $r_i \geq 1$ for $i = 1, \ldots, 5$. Note that $G \setminus C_5$ is isomorphic to the forest $\cup_{i=1}^{5} S_{r_i+1}$. From Theorem 2.1, $T_2(G) \leq T_2(C_5) + T_2(\cup_{i=1}^{5} S_{r_i+1}) \leq e(C_5) + 3 + r_1 + 1 + r_2 + 1 = r_1 + r_2 + 10 < e(G) + 3$. If there exists one vertex u_j such that $r_j = 0$ and $r_i \geq 1$ for every $i \neq j$, the proof is identical to the previous case.

(ii) Assume that $r_1 = n - 5$ and $r_i = 0$ for each $i = 2, \ldots, 5$. From Theorem 2.1, $T_2(G) \leq T_2(S_{n-4} \cup 2K_2) + T_2(S_3 \cup K_2) = n + 3 = e(G) + 3$.

(iii) Assume that there exist two vertices u_i and u_j such that $r_i \geq r_j \geq 1$ and $r_s = 0$ for $s \neq i, j$. Note that $G \setminus C_5$ is isomorphic to $S_{r_i+1} \cup S_{r_j+1} \cup 3K_1$. Next, we consider the two possible subcases, that is, $j = i + 1$ and $j = i + 2$.

(a) Let $j = i+1$. If $r_j = 1$, from Theorem 2.1, $T_2(G) \leq T_2(S_{r_i+1} \cup S_3 \cup K_2) + T_2(S_3 \cup K_2) = (r_i+1+3)+(3+2) = (5+r_i+1)+3 = n+3$. If $r_j \geq 2$. From Theorem 2.1, $T_2(G) \leq T_2(S_{r_i+1} \cup S_{r_j+1} \cup S_3) + T_2(P_4) \leq (r_i+r_j+2)+6 = (r_i+r_j+5)+3 = n+3$.

(b) Let $j = i+2$. Consider the graphs G_1 and G_2 that are isomorphic to S_{r_i+2} and S_{r_j+3}, respectively. Observe that $G_1 \approx G_2$ is isomorphic to G and both satisfy Conjecture 1.2 since they are trees. From Lemma 2.6, $T_2(G) = T_2(G_1 \approx G_2) \leq e(G_1 \approx G_2) + 3 = e(G) + 3$;

(iv) Consider that only r_i, r_j and r_t are non-zero such that $r_i \geq r_j \geq r_t \geq 1$. There are only two possibilities:

(a) Let $j = i + 1$ and $t = i + 2$. In this case, the subgraph $G \setminus C_5$ is isomorphic to $S_{r_i+1} \cup S_{r_{i+1}+1} \cup S_{r_{i+2}+1} \cup 2K_1$. From Theorem 2.1, $T_2(G) \leq T_2(S_{r_i+1} \cup S_{r_{i+1}+1} \cup S_{r_{i+2}+1} \cup K_2) + T_2(P_5) < (r_i + r_{i+1} + 2) + (4 + 3) \leq n + 3 = e(G) + 3$.

(b) Let $j = i+1$ and $t = i+3$. In this case, the subgraph $G \setminus C_5$ is isomorphic to $S_{r_i+1} \cup S_{r_{i+1}+1} \cup S_{r_{i+3}+1} \cup 2K_1$. From Theorem 2.1, $T_2(G) \leq T_2(S_{r_i+2} \cup S_3) + T_2(S_{r_{i+1}+2} \cup S_{r_{i+3}+1} \cup K_2) \leq (r_i+2+3)+(r_{i+1}+1+r_{i+3}+2) = n+3 = e(G)+3$.

Case (C): Assume $g = 7$ and let $V(C_7) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$.

(i) Consider that there exist $t \geq 3$ at least three vertices such that $r_i \neq 0$, say $r_i \geq r_j \geq \ldots \geq r_t$. The subgraph $G \setminus C_7$ is isomorphic to $\left(\bigcup_{i=1}^{t} S_{r_i+1}\right) \cup (7-t)K_1$. From Theorem 2.1, $T_2(G) \leq T_2(C_7) + T_2\left(\left(\bigcup_{i=1}^{t} S_{r_i+1}\right) \cup (7-t)K_1\right) < 8 + (r_i+1+r_j+1) < n+3$.

(ii) Consider that there exist u_i such that $r_i = n-7$. Applying Theorem 2.1, we get $T_2(G) \leq T_2(S_{n-6} \cup 3K_2) + T_2(S_3 \cup 2K_2) = (n-6+2)+(3+2) = n+1 < n+3$.

(iii) Consider that there exist u_i and u_j such that $r_i \geq r_j \geq 1$ and $r_i + r_j = n-7$. Note that $G \setminus C_7$ is isomorphic to $S_{r_i+1} \cup S_{r_j+1} \cup 5K_1$. From Theorem 2.1 we get $T_2(G) \leq T_2(C_7) + T_2\left(S_{r_i+1} \cup S_{r_j+1} \cup 5K_1\right) < 8 + r_i + r_j + 2 = n+3$.

These cases complete the proof. ■

Lemma 3.7 *If G is a unicyclic graph of order $n \geq 4$ with girth $g \in \{3,5,7\}$ that has not only pendant edges attached to the vertices of the cycle. Then $T_2(G) \leq e(G)+3$.*

Proof Suppose that G is a unicyclic graph of order $n \geq 4$ with girth $g \in \{3,5,7\}$ that has not only pendant edges attached to the vertices of the cycle. Consider each connected component H_i of the graph $G - C_g$ for $i = 1, \ldots, p$ that is a tree and the graphs $H_1 \sim C_g, (H_1 \sim C_g) \sim H_2, \ldots, ((H_1 \sim C_g) \sim H_2) \sim \ldots \sim H_p$. From Lemmas 2.5 and 3.6, inequality of the Conjecture 1.2 holds for each of the previous graphs. Since the last graph is isomorphic to G, the result follows. ■

Now, our main result follows from Lemmas 3.4, 3.5, 3.6 and 3.7.

Theorem 3.8 *If G is a unicyclic graph of order n then $T_2(G) \leq e(G)+3$.*

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