Optimality conditions for satellite module layout design problem

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ABSTRACT

In this paper we present a sufficient condition for weakly efficient solutions for the multiobjective nonidentical circle packing problem, in which none of the pairs of circles overlap. We considered two important objectives for the allocation of instruments or pieces of equipment in a spacecraft or satellite: (1) to minimize the radius of the circle containing all of the objects and (2) to minimize the imbalance. To demonstrate these results we also established new optimality conditions for the vector optimization problem.
Keywords: Multiobjective nonidentical circle packing problem; Layout optimization problem; Optimality conditions; Weakly efficient solutions

RESUMO

Nesse artigo apresentamos uma condição suficiente para soluções fracamente eficientes em um problema multiobjetivo de empacotamento de círculos, sem sobreposição. Nós consideramos dois importantes objetivos para alocação de instrumentos em um satélite ou foguete: (1) minimizar o raio do recipiente circular que contém todos os objetos e (2) minimizar o desequilíbrio. Para demonstrar os resultados foram estabelecidos novas condições de otimalidade para um problema de otimização vetorial.
Palavras-chave: Problema de empacotamento de cículos; Problema de Otimização de Layout; Condições de otimalidade; Soluções fracamente eficientes

1 Introduction

The satellite module layout design problem, aldo called multiobjective nonidentical circle packing problem (MoCPP) [1], aims to obtain efficient solutions to two objectives: reducing the radius of the container and the imbalance, seeking the allocation of instruments or pieces of equipment in a spacecraft or satellite, without overlapping.

We presented bellow a mathematical model for (MoCPP) [1] [2]. We assume there are n objects to be laid out. The radius of the objects are $r_1, r_2, \ldots, r_n$, and their masses are $m_1, m_2, \ldots, m_n$. The radius of the container is r_0. The origin of the Cartesian coordinate system is set to the center of the container. Let the $2n+1$-dimension vector $z = (r, x_1, , x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ denote a layout, where (x_i, y_i) is the center of object i. Therefore the mathematical model is:

$$(MoCPP)\left\{\begin{array}{c} z^* = min_z\ f(z) = (f_1(z), f_2(z)) \\ \text{s. t. } z \in X \end{array}\right.$$

where

- $f_1(z) = r \equiv$ The envelopment objective attempts to minimize the radius of the circle containing all of the objects.
- $f_2(z) = \left(\sum_{i=1}^n m_i x_i\right)^2 + \left(\sum_{i=1}^n m_i y_i\right)^2 \equiv$ The imbalance objective attempts to minimize the non-equilibrium of mass for a collection of objects around the central point.
- $X = \{z \in R^{2n+1}, r \leq r_o;\quad r \geq r_i$ and $x_i^2 + y_i^2 \leq r^2 - 2rr_i + r_i^2$ for all $i = 1, \ldots, n;\ (x_i - x_j)^2 + (y_i - y_j)^2 \geq r_i^2 + 2r_ir_j + r_j^2$ for all $i = 1, \ldots, n-1;\ j = i+1, i+2, \ldots, n.\}$.

Despite being a vector optimization problem, in the literature we have only found scalar approaches, i.e. the problem is treated as a scalar optimization problem (with only one objective function) [1]. As a consequence, we have not found papers that analyze or demonstrate any optimality conditions for this multiobjective problem. However, the same scalar circle packing problem (CPP) is a difficult problem to solve: it has an infinite number of alternative optima, an exponential number of local optima which are not globally optimal, and an uncountable set of stationary points i.e. solutions that satisfy the KKT conditions without being local optima [3]. Castillo et al. [4] presented several industrial applications of the circle packing problem and some exact and heuristic strategies for their solution.

The outline of this paper is as follows: Section 2 gives a general introduction of the multiobjective optimization problem and presents a characterization of the KT-pseudoinvex-III function. Section 3 presents a sufficient condition for weakly efficiency for the multiobjective nonidentical circle packing problem. Finally, we state some final conclusions in section 4.

2 Optimality conditions for Vector Optimization Problem

The following convention for equalities and inequalities is assumed below: if $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, then $x = y \Leftrightarrow x_i = y_i, \quad \forall i = 1, \ldots, n$; $x < y \Leftrightarrow x_i < y_i, \quad \forall i = 1, \ldots, n$; $x \leqq y \Leftrightarrow x_i \leq y_i, \quad \forall i = 1, \ldots, n$; $x \leq y \Leftrightarrow x_i \leq y_i, \quad \forall i = 1, \ldots, n$ and there exist$j$ such that $x_j < y_j$. Similarly, $>, \geqq, \geq$.

We will work with the following vector optimization problem (VOP):

$$
\textbf{(VOP)} \quad \begin{array}{ll} Min & f(x), \\ s.t.: & g(x) \leqq 0, \\ & x \in S \end{array}
$$

where S is an open subset of $\mathbb{R}^n$, $f = (f_1, \ldots, f_p) : S \subseteq \mathbb{R}^n \to \mathbb{R}^p$ and $g = (g_1, \ldots, g_m) : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are differentiable.

We can now define the concepts of efficient solution and weakly efficient solution for (VOP)

Definition 1. *A feasible point $\bar{x}$ is said to be an efficient solution for (VOP) if there does not exist another feasible point, x, such that $f(x) \leq f(\bar{x})$.*

Definition 2. *A feasible point $\bar{x}$ is said to be a weakly efficient solution for (VOP) if there does not exist another feasible point, x, such that $f(x) < f(\bar{x})$.*

It is easy to see that any efficient solution is a weakly efficient solution.

Definition 3. *A feasible point $\bar{x}$ for (VOP) is said to be a Kuhn-Tucker vector critical point(KTVCP) if there exist $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^m$ such that*

$$\lambda^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}) = 0 \tag{1}$$

$$\mu^T g(\bar{x}) = 0 \tag{2}$$

$$\mu \geqq 0 \tag{3}$$

$$\lambda \geq 0 \tag{4}$$

Osuna et al. [5] proved necessary and sufficient conditions for weakly efficient solutions through Kuhn-Tucker optimality conditions. This search is resolved by the class KT-pseudoinvex-I, which we defined in the following way.

Definition 4. *The (VOP) is said to be KT-pseudoinvex-I if there exists a vector function $\eta : S \times S \to \mathbb{R}^n$ such that for all feasible points $x, \bar{x}$*

$$
f(x) - f(\bar{x}) < 0 \Rightarrow \begin{cases} \nabla f(\bar{x})\eta(x,\bar{x}) < 0 \\ \nabla g_j(\bar{x})\eta(x,\bar{x}) \leqq 0, \quad \forall j \in I(\bar{x}) \end{cases}
$$

where $I(\bar{x}) = \{j = 1, \ldots, m : g_j(\bar{x}) = 0\}$.

Osuna et al. [5] established the following characterization theorem for (VOP).

Theorem 1. *Every KTVCP is a weakly efficient solution of (VOP) if and only if problem (VOP) is KT-pseudoinvex-I.*

Arana et al. [6] have studied the set of efficient solutions for (VOP), which are contained into the set of weakly efficient solutions. For that purpose, they have proposed the KT-pseudoinvexity-II. They have proved that KT-pseudoinvexity-II is necessary and sufficient for a Kuhn-Tucker point to be an efficient solution for (VOP). However, we focus our study on the weakly efficient solutions for (MoCPP), for which we need new conditions on the functions involved. To obtain a sufficient condition for a solution to be weakly efficient for (MoCPP) we need the following definitions and the subsequent results.

Definition 5. *A feasible point $\bar{x}$ for (VOP) is said to be a strict Kuhn-Tucker vector critical point, SKTVCP, if there exist $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^m$ such that*

$$\lambda^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}) = 0 \tag{5}$$

$$\mu^T g(\bar{x}) = 0 \tag{6}$$

$$\mu \geqq 0 \tag{7}$$

$$\lambda > 0 \tag{8}$$

It is easy to see that every SKTVCP is KTVCP.

Definition 6. *A (VOP) is said to be KT-pseudoinvex-III at $\bar{x}$ if there exists a vector function $\eta : S \times S \to R^n$ such that for all feasible points x*

$$f(x) - f(\bar{x}) < 0 \Rightarrow \begin{cases} \nabla f(\bar{x})\eta(x, \bar{x}) \leq 0 \\ \nabla g_j(\bar{x})\eta(x, \bar{x}) \leqq 0, \quad \forall j \in I(\bar{x}) \end{cases}$$

where $I(\bar{x}) = \{j = 1, \ldots, m : g_j(\bar{x}) = 0\}$.

Theorem 2. *If x^* is a SKTVCP and (VOP) is KT-pseudoinvex-III at x^* then x^* is a weakly efficient solution of (VOP).*

Proof:
Let x^* be a SKTVCP and (VOP) KT-pseudoinvex-III. Suppose that there exists a feasible $\bar{x}$ such that $f(\bar{x}) < f(x^*)$. In this case $f(\bar{x}) - f(x^*) < 0$. Since (VOP) is KT-pseudoinvex-III there exist $\eta : S \times S \to R^n$ such that:

$$\nabla f(x^*)\eta(\bar{x}, x^*) \leq 0$$

$$\nabla g_j(x^*)\eta \leqq 0, \quad \forall j \in I(x^*)$$

where $I(x^*) = \{j = 1, \ldots, m : g_j(x^*) = 0\}$.

Therefore, there exists $1 \leq k \leq p$ such that $\nabla f_k(x^*)\eta(\bar{x}, x^*) < 0$.

Since x^* is SKTVCP, there exist $\lambda > 0$ and $\mu \geqq 0$ such that:

$$\sum_{k=1}^{p} \lambda_k \nabla f_k(x^*) + \sum_{i \in I(x^*)} \mu_i \nabla g_i(x^*) = 0$$

However, we have $\lambda_k \nabla f_k(x^*)\eta(\bar{x}, x^*) < 0$ and $\mu_i \nabla g_i(x^*)\eta(\bar{x}, x^*) \leq 0$ for $i \in I(x^*)$. Therefore:

$$\lambda^t \nabla f(x^*)\eta(\bar{x}, x^*) + \mu_{I(x^*)} \nabla g_{I(x^*)}(x^*)\eta(\bar{x}, x^*) < 0$$

which leads to contradiction. □.

We say that (VOP) is KT-pseudoinvex-III if (VOP) is KT-pseudoinvex-III at x for all $x \in X$.

Theorem 3. *Every SKTVCP is a weakly efficient solution of (VOP) if and only if (VOP) is KT-pseudoinvex-III.*

Proof:
i) [$\Rightarrow$] Let us suppose that there exist two feasible points $\bar{x}$ and x^* such that

$$f(\bar{x}) - f(x^*) < 0$$

since otherwise (VOP) would be KT-pseudoinvex-III, and the result would be proved. This means that x^* is not a weakly efficient solution, and by using the initial hypothesis, x^* is not a SKTVCP, i.e.,

$$\lambda^T \nabla f(x^*) + \mu^T \nabla g_{I(x^*)}(x^*) = 0$$

has no solution $\lambda > 0$ and $\mu \geqq 0$. Therefore, by Tucker's theorem (Mangasarian 1969), the system:

$$\begin{aligned} \nabla f(x^*)\eta(\bar{x}, x^*) &\leq 0 \\ \nabla g_j(x^*)\eta(\bar{x}, x^*) &\leqq 0, \quad \forall j \in I(x^*) \end{aligned}$$

has the solution $\eta(\bar{x}, x^*) \in R^n$, where $I(x^*) = \{j = 1, \dots, m : g_j(x^*) = 0\}$.
ii) [$\Leftarrow$] The proof is similar to the one of theorem 2. $\square$.

Thus, we get an important characterization that will be used in the next section to obtain a sufficient condition for a solution to be weakly efficient for (MoCPP).

3 Sufficient condition for weak efficiency

We show below that (MoCPP) is KT-pseudoinvex-III.

Theorem 4. *If z^* is a feasible point for (MoCPP) then (MoCPP) is KT-pseudoinvex-III at z^*.*

Proof:

Let z^* is a feasible point for (MoCPP). We have:

$$\nabla f_1(z^*) = (1, 0, \dots, 0)$$

$$\nabla f_2(z^*) = (0, 2m_1 \sum_{i=1}^n m_i x_i^*, 2m_2 \sum_{i=1}^n m_i x_i^*, \dots, 2m_n \sum_{i=1}^n m_i x_i^*, 2m_1 \sum_{i=1}^n m_i y_i^*, 2m_2 \sum_{i=1}^n m_i y_i^*, \dots, 2m_n \sum_{i=1}^n m_i y_i^*)$$

$$\nabla g_k(z^*) = \begin{cases} (1, 0, \dots, 0) & \text{if } k = 1 \\ (-1, 0, \dots, 0) & \text{if } k = 2, \dots, n+1 \\ (-2r^* + 2r_i, 0, \dots, 0, 2x_i^*, 0, \dots, 0, 2y_i^*, 0, \dots, 0) & \text{if } k = n+1+i \end{cases}$$

and for $k = 1 + 2n + l$, $l = 1, \dots n(n-1)/2$ we have:

$$\nabla g_k(z^*) = (0, \dots, 0, 2(x_i^* - x^* - j), 0, \dots, 0, 2(y_i^* - y^* j), 0, \dots, 0, -2(y_i^* - y^8 j), 0, \dots, 0)$$

Let $\bar{z} \in X$ such that $f(\bar{z}) < f(z^*)$.
We must find $\eta(z*, \bar{z}) \in \mathbb{R}^{2n+1}$ such that:

$$\nabla f(\bar{z})\eta(z*,\bar{z}) < 0$$
$$\nabla g_j(\bar{z})\eta(z*,\bar{z}) \leqq 0, \quad \forall j \in I(\bar{\bar{z}})$$

Since $f_2(\bar{z}) < f_2(z^*)$, we have $f_2(z^*) > 0$. Therefore $\sum_{l=1}^{n} m_l x_l \neq 0$ or $\sum_{l=1}^{n} m_l y_l \neq 0$. Without loss of generality we can assume that $\sum_{l=1}^{n} m_l x_l \neq 0$.

We take $\eta(z*,\bar{z}) = \{(\eta_i),\ i = 1, 2, \ldots, 2n+1$ such that:

- $\eta_1 = 0$;
- $\eta_i = \eta_{n+1+i} = 0$ for $i = 2, \ldots, n$ such that $r^* = r_i + \sqrt{x_i^2 + y_i^2}$, otherwise we take, for $i = 2, \ldots n+1$ $\eta_i = 1$, if $\sum_{l=1}^{n} m_l x_l^* > 0$ and $\eta_i = -1$ if $\sum_{l=1}^{n} m_l x_l^* < 0$.
- Similarly we take for $i = n+2, \ldots 2n+1$ $\eta_i = 1$ if $\sum_{l=1}^{n} m_l y_l^* > 0$ and $\eta_i = -1$ if $\sum_{l=1}^{n} m_l y_l^* < 0\}$.

We have:

$$\nabla f_1(z^*)\eta(z*,\bar{z}) = \nabla g_i(z^*)\eta(z*,\bar{z}) = \nabla g_k(z^*)\eta(z*,\bar{z}) = 0$$

for $k = 1, \ldots, n+1$, since $\eta_1 = 0$. We also have:

$$\nabla f_2(z^*)\eta(z*,\bar{z}) = -2(\sum_{k\in K} m_k)|\sum_{i=i}^{n} m_i x_i| - 2(\sum_{k\in K} m_k)|\sum_{i=i}^{n} m_i y_i| < 0$$

where $i, j \in K$ if $r > r_i + \sqrt{x_i^2 + y_i^2}$ for all $i = 1, \ldots, n$; $j = i+1, i+2, \ldots, n$.

Finally, if $k = 1 + n + l$, $l = 1, \ldots n(n-1)/2$ we have:

$$\nabla g_k(z^*)\eta(z*,\bar{z}) = 2(x_i^* - x_j^*) - 2(x_i^* - x_j^*) + 2(y_i^* - y_j^*) - 2(y_i^* - y_j^*) = 0$$

$\square$.

Thus, we have the following sufficient condition for weak efficiency in a (MoCPP):

Corollary 1 (Sufficient condition for Weak Efficiency)**.** *Let $x^* \in X$. If x^* is a SKTVCP for (MoCPP) then x^* is a weakly efficient solution for (MoCPP).*

4 Conclusion

In this paper we present a sufficient condition for weakly efficient solutions for the multiobjective nonidentical circle packing problem (MoCPP). For this purpose, we also obtain an important characterization of the problems for which every strict Kuhn-Tucker vector critical point is a weakly efficient solution, and that can be useful in other vector optimization problems.

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